Quantum Theory of Light (WS17/18)

Exercises 2

(For exercise class on Thu, December 14th 2017.)

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1. **Hong–Ou–Mandel effect for higher field excitations.**
   Consider the Hong–Ou–Mandel interferometer where in-going modes are in the number states $|n\rangle_0 |n\rangle_1$.
   (a) Compute the state $|\psi\rangle_{2,3}$ of the outgoing modes.
   (b) Write the result as $|\psi\rangle_{2,3} = \sum_{k,l} c_{k,l} |k\rangle_2 |l\rangle_2$. What is the rank of the matrix $(c_{k,l})$?
   (c) Proceed similarly for coherent in-going states, $|\alpha\rangle_1 |\beta\rangle_2$.

2. **Time correlations in the Jaynes–Cummings model.**
   Compute the correlation of the dipole operator at different times for the atom in the excited state $|e\rangle$ and the field mode in a coherent state $|\alpha\rangle$.
   Instructions.
   The Jaynes–Cummings Hamiltonian in the interaction picture is given by

$$H(t) = \hbar\nu a^\dagger a + \frac{\hbar}{2}\omega\sigma_z + \hbar\beta(\sigma_+ a \mathrm{e}^{i\Delta t} + a^\dagger \sigma_- \mathrm{e}^{-i\Delta t}),$$

   where $\Delta = \omega - \nu$, $\sigma_+ = |e\rangle\langle g| = \sigma_-^\dagger$, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$.
   (a) How is the relation between β and the dipole moments $\mathrm{tr}(\sigma_\pm \hat{d})$? (Hint: The interaction term term in the Hamiltonian corresponds to $-\hat{d}\cdot\hat{E}$.)
   (b) Transform $H(t)$ to a form that is not explicitly time-dependent.
   (c) Compute the Heisenberg equations for a, σ_-, σ_z, and $N = a^\dagger a + \sigma_+\sigma_-$.
   (d) Find a second constant of motion C of the form $C = c_1\sigma_z + c_2\sigma_+ a + \text{h.c.}$
   (e) Compute

$$-i\left(\frac{\Delta}{2} + C\right)\dot{\sigma}_- + \left(\nu C - \frac{\Delta^2}{2} + \frac{\omega}{2}\Delta\right)\sigma_-$$

   and use the result to obtain

$$C_2\ddot{\sigma}_- + C_1\dot{\sigma}_- + C_0\sigma_- = 0,$$

   where C_2, C_1, and C_0 are constant operators.
   (f) Solve the above differential equation and compute $\langle a,\alpha|\sigma_+(t)\sigma_-(t+\tau)|a,\alpha\rangle$.

3. **Dressed states.**
   Consider the Jaynes–Cummings Hamiltonian, where the interaction term is replaced by

$$H_I = \hbar\beta a^\dagger a(\sigma_+ + \sigma_-).$$

   (a) Obtain the dressed states for this model.
   (b) Compute the inversion $W(t) = |\langle e,\psi(t)\rangle| - |\langle g,\psi(t)\rangle|$ for $|\psi(0)\rangle = |g,\alpha\rangle$, i.e., an initial state where the atom is in the ground state and the photon field is in a coherent state.

4. **Revival of the atom excitation.**
   Using numerical methods, play with the inversion $W(t)$ for the Jaynes–Cummings model, when the initial state of the field is coherent and the initial state of the atom is excited. Demonstrate that, at first, the excited level decays, but later it enjoys a revival. (Hint: Use $\Delta = 0$ and the time unit $1/\beta$.)