Quantum theory of light

Lecturer: Matthias Kleinmann (Tue 14:15, Room B030)
Exercises: Chau Nguyen (Mon 16:15, Room D120) **Sheet 1**

Hand in: Tue 22.10.2019 *(questions marked as $*$ are optional)*
Discussion date: Mon 28.10.2019

1. Review of the quantum harmonic oscillators

Consider the quantum harmonic oscillator with Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 \tag{1}$$

where X and P are the position and momentum operators, $[X, P] = i\hbar$. The creator and annihilator are defined by

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}),\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega}), \tag{2}$$

then $[a, a^\dagger] = 1$ and one can write $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. The energy eigenbasis is denoted by $|n\rangle$, $n = 0, 1, \ldots$

(a) [5pts] Suppose the system is initially in state $|\psi(0)\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. In the Schrödinger picture, find the state $|\psi(t)\rangle$ of the system at time t expressed in the energy eigenbasis.

(b) [5pts] Compute $X(t)$ and $P(t)$ in the Heisenberg picture.

(c) [5pts] Compute $\langle X(t)X(0)\rangle$ for the initial state $|\psi(0)\rangle$.

2. Vacuum fluctuations of the electromagnetic field

Recall from the lecture that the electromagnetic field linearly polarised along the x-axis, propagating with wavevector k along the z-axis, is given by

$$E_x^k(z,t) = E_0^k[a_k(t) + a_k^\dagger(t)]\sin(kz),\ B_y^k(z,t) = B_0^k[a_k(t) + a_k^\dagger(t)]\cos(kz), \tag{3}$$

with $\omega = kc$, $E_0^k = \sqrt{\hbar\omega/\epsilon_0 V}$, $B_0^k = \mu_0/k\sqrt{\epsilon_0\hbar\omega^3/V}$, where V is volume of the cavity containing the field. Confined to the cavity of length L, the wavevector is quantised as $k = n\pi/L$, $n = 1, 2, \ldots$

(a) [5pts] Compute the fluctuation $\left\langle[\Delta E_x^k(z,t)]^2\right\rangle = \left\langle[E_x^k(z,t)]^2\right\rangle - \left\langle E_x^k(z,t)\right\rangle^2$ for the vacuum of the mode.

(b) [5pts] Consider all modes in the cavity, the total field operators are then

$$E_x(z,t) = \sum_k E_x^k(z,t),\ B_y(z,t) = \sum_k B_x^k(z,t), \tag{4}$$

Show that the total fluctuation is the sum of the fluctuations of all the modes.

Remark: For simplicity, we do not consider waves propagating in other directions, and fix the polarisation.

(c) [5pts] Estimate the physical fluctuations a physical probe can detect when placed in the middle of the cavity. Note that an actual physical probe always has finite size, say δ ($\delta \ll L$).