Quantum Information Theory
Exercise sheet 1

Lecture: Prof. Dr. Otfried Gühne   Exercise: Chau Nguyen & Xiao-Dong Yu
Lecture: Wednesday, 12:00, Room D115; Friday, 12:30, Room B030
Exercise: Friday, 14:15, Room D120

1. **Positive semidefinite matrices**
   Show that for hermitean 2×2-matrices $A = (a_{ij})$ the following statements are equivalent:
   (a) A has no negative eigenvalues;
   (b) $\langle\psi|A|\psi\rangle \geq 0$ for all vectors $|\psi\rangle$;
   (c) $\det A \geq 0$, $a_{11} \geq 0$ and $a_{22} \geq 0$.

2. **Bloch vectors**
   Any single-qubit density matrix ρ can be parameterized in terms of the Pauli matrices by a Bloch vector $\vec{a}$:
   $$\rho = \frac{1}{2}\big(\mathbf{1} + \sum_{i=1}^{3} a_i \sigma_i\big).$$
   On the other hand, ρ can be expressed in its eigenbasis as
   $$\rho = p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|$$
   with $0 \leq p \leq 1$.
   (a) Determine $\det\rho$. Which condition on $\vec{a}$ results from $0 \leq p \leq 1$? Describe the set of all density matrices geometrically.
   (b) Describe a minimal set of observables, such that from the corresponding mean values $\vec{a}$ and therefore ρ can be reconstructed.
   (c) Determine p as a function of $\vec{a}$, and compute the eigenvectors $|\phi_i\rangle$ as a function of $\vec{a}$.

3. **Schmidt decomposition**
   Determine the Schmidt decomposition (in (c) and (d) for all bipartitions) of the following states (this can be done without long computation):
   (a) $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$
   (b) $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$
   (c) $(|000\rangle + 2|011\rangle + |110\rangle)/\sqrt{6}$
   (d) $(|000\rangle + \mathrm{i}|010\rangle + \mathrm{i}|101\rangle - |111\rangle)/2$