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Measurement incompatibility: a resource for quantum steering

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"Physics is like sex. Sure, it may have some practical results, but that's not why we do it."

Richard P. Feynman

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Contents

Dedicated to my Dad.

Abstract

One of the most prominent features of quantum mechanics is the incompatibility of measurements. This feature, not present in classical physics, captures the fact that when performing observations on quantum systems we find that, in general, the results of the subsequent measurement of different (and classically uncorrelated) properties depend on the order on which the measurements are carried out.

In this work, measurement incompatibility is regarded as an useful property of experimental setups, for example, recent results [1] have shown that incompatibility of measurements is needed for the task of quantum steering. After an introduction to selected notions of quantum theory needed in the rest of the discussion (chapter 1), chapter 2 formalizes the notion of measurement incompatibility for various objects used to describe measurement procedures. Chapter 3 offers an introduction to quantum steering and presents steering quantifiers often used in the literature. Chapter 4 explores the connection between the steerability of a setup and the compatibility of observables, and presents original results on the steerability of broadcast quantum channels. Finally, in chapter 5 a formal resource theory for incompatibility is developed, joining partial results present in the literature.

Chapter 1

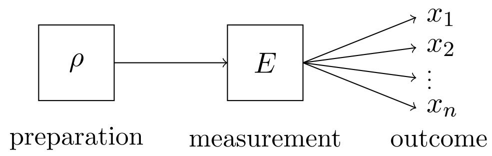
Introduction

The aim of this chapter is to provide a brief introduction to the mathematical notions and formalism used in the rest of this work, as well as giving physical reasons for the need of such formalism.

1.1 Basic notions

1.1.1 Quantum mechanics as a probabilistic theory

Physical theories are formal devices used to predict the result of a measurement procedure on a physical system. Such procedure usually consists of a preparation stage in which the system undergoes a controlled evolution and a subsequent interaction with a measurement device, which outputs one of the possible outcomes of the experiment. This setup is depicted in figure:



A probabilistic theory makes predictions in the form of probabilities; *i.e.*, it assigns probabilities $p(x_i|\rho, E)$ to each experimental outcome x_i, given the preparation procedure ρ and the measurement E. Quantum mechanics is a probabilistic theory built on *Hilbert spaces*, that is, operators on *Hilbert spaces* are used to construct the 'phase space' of the theory.

1.1.2 States as density operators

Preparation procedures (that is, 'states' of the system) are formalized by the use of unit trace, positive[1] operators acting on a Hilbert space $\mathcal{H}$. Those operators are called *density operators*. We will denote the space of density operators on a Hilbert space $\mathcal{H}$ with $S(\mathcal{H})$.
This definition of states differs from the textbook notion in which states are vectors in $\mathcal{H}$; the main reason to introduce such operators is to be able to unify the formal treatment of pure states and ensambles of states, as will be clear in the next section. With this definition, the notion of pure states can be recovered as extremal states in the convex[2] space $S(\mathcal{H})$, that is, the density operator for a pure state $|\psi\rangle$, given by $\rho = |\psi\rangle\langle\psi|$, cannot be written as a convex combination of two other operators. Note that, since $|\psi\rangle\langle\psi|$ is a projector, we have $\rho^2 = \rho$.
For statistical mixtures of states, the density operator is defined as $\rho =$

[1]An operator V is said to be positive if $\langle\psi| V |\psi\rangle \geq 0 \;\; \forall |\psi\rangle \in \mathcal{H}$

[2]The convexity of $S(\mathcal{H})$ is trivial: any convex combination of positive, unit-trace operators is also positive and self-adjoint. The trace condition is also trivially satisfied.

$\sum_i p_i |\psi_i\rangle \langle\psi_i|$, where p_i is the probability for the system to be in the state $|\psi_i\rangle$. Since $\sum_i p_i = 1$, the condition $\mathrm{tr}[\rho] = 1$ is satisfied. However, it's easy to check that the relation $\rho^2 = \rho$ holds only if ρ describes a pure state.

1.1.3 Multipartite systems

In the following discussion, we will often need to formalize the notion of multipartite state, that is, the state of a large system composed of multiple subsystems. We denote the Hilbert space of each subsystem with $\mathcal{H}_1, \mathcal{H}_2, \ \ldots \mathcal{H}_N$, and describe the larger system as the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$. A basis for $\mathcal{H}$ can be obtained as $\{|\varphi_{1,i}\rangle \otimes |\varphi_{2,j}\rangle \otimes \cdots \otimes |\varphi_{N,l}\rangle\}$, where $\{|\varphi_{1,i}\rangle\}$ is a basis for $\mathcal{H}_1$, and so on.
Restricting ourselves to only two subsystems, in order to ease the notation, we see that a general vector can be decomposed on this basis as:

$$|\psi\rangle = \sum_{i,j} \alpha_{i,j} |\varphi_{1,i}\rangle \otimes |\varphi_{2,j}\rangle \tag{1.1}$$

Therefore, density operators on $\mathcal{H}$ take the form:

$$\rho = \sum_{i,j,k,l} \alpha_{i,j}\alpha_{k,l} |\varphi_{1|i}\rangle \langle\varphi_{1|k}| \otimes |\varphi_{2|j}\rangle \langle\varphi_{2|l}| \tag{1.2}$$

Theorem 1.1. *(Schmidt decomposition) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces of dimension n and m respectively; w.l.o.g we can assume $m \leq n$. For any vector $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there exists orthonormal basis $\{|\varphi_{1,i}\rangle\}$, $\{|\varphi_{2,i}\rangle\}$*

for $\mathcal{H}_1$ and $\mathcal{H}_2$ and non-negative reals α_i such that:

$$|\psi\rangle = \sum_{i}^{m} \alpha_i |\varphi_{1,i}\rangle \otimes |\varphi_{2,i}\rangle \tag{1.3}$$

The number of non-zero coefficients α_i is known as the Schmidt rank of $|\psi\rangle$

Proof. The proof of this fact can be found in [2]. □

Definition 1.2. Given a multipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we say that the state is *entangled* if its Schmidt rank is bigger than 1. Otherwise we say that the state is separable.

Using the density operator notation for states, we have that if the total larger system has a density matrix ρ_{12}, the marginal density matrix for one subsystem can be obtained by taking a partial trace over the other subsystem. That is:

$$\rho_1 = \mathrm{tr}_2[\rho_{12}] = \sum_i \langle\varphi_{2,i}| \rho_{12} |\varphi_{2,i}\rangle \tag{1.4}$$

Proposition 1.3. *If a bipartite state ρ_{AB} has a pure marginal state ρ_A, then the state ρ_{AB} is separable.*

Proof. If the bipartite state is pure, the density matrix ρ_{AB} can always be written as:

$$\rho_{AB} = |\psi\rangle\langle\psi| = \sum_{i,j} \alpha_i \alpha_j |\varphi_{A,i}\rangle\langle\varphi_{A,j}| \otimes |\varphi_{B,i}\rangle\langle\varphi_{B,j}| \tag{1.5}$$

where α_i are the Schmidt coefficient of $|\psi\rangle$ as in Theorem 1.1.
We can now write:

$$\begin{aligned}\rho_A = \mathrm{tr}_B[\rho_{AB}] = \sum_l \langle\varphi_{B,l}|\psi\rangle\langle\psi|\varphi_{B,l}\rangle = \\ = \sum_{l,i,j} \alpha_i\alpha_j\,|\varphi_{A,i}\rangle\langle\varphi_{A,j}| \otimes \langle\varphi_{B,l}|\varphi_{B,i}\rangle\langle\varphi_{B,j}|\varphi_{B,l}\rangle = \\ = \sum_{l,i,j} \alpha_i\alpha_j\,|\varphi_{A,i}\rangle\langle\varphi_{A,j}|\,\delta_{li}\delta_{lj} = \\ = \sum_l \alpha_l^2\,|\varphi_{A,l}\rangle\langle\varphi_{A,l}|\end{aligned} \tag{1.6}$$

If ρ_A is pure at most one of the α_l can be non-zero, therefore he have that the Schmidt rank of $|\psi\rangle$ is one, and that the state is separable.
In the case of mixed states, it's easy to see that a mixed state can have a pure marginal only if it is separable; that is, only if the total state is a tensor product of a pure state with a mixed one. □

1.1.4 Qubits

Throughout this work, we will often use the notion of qubits in proof and examples. A qubit is any physical quantum system that can be described by a 2-dimensional Hilbert space; the simplest example of a qubit is a spin-$\frac{1}{2}$ particle.
The states are represented as two component vectors, represented in coordinates with respect to the standard basis $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. It is customary to extend the notion of Pauli matrices, usually used for spin

systems, to any kind of two-level system, defining the operators:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.7}$$

The vectors $|0\rangle$ and $|1\rangle$ are eigenvectors of σ_z, while the eigenvectors of σ_x and σ_y are respectively $|\pm\rangle = |0\rangle \pm |1\rangle$ and $|\leftrightarrows\rangle = |0\rangle \pm i\,|1\rangle$.

Bloch sphere representation

An arbitrary density operator ρ representing the state of a qubit system is any positive operator such that $\mathrm{tr}[\rho] = 1$. We can parametrize this class of operators by the following relation[2]:

$$\rho = \frac{1}{2}\left(\mathbb{1} + \lambda \hat{n} \cdot \vec{\sigma}\right) \tag{1.8}$$

where $\hat{n}$ is a unit vector in $\mathbb{R}^3$, $\lambda \in [0, 1]$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. With this in mind, we can represent all qubit states as the vectors $\vec{v} = \lambda\hat{n}$ contained in a unit sphere; this representation is known as the Bloch sphere representation. Note that the points on the surface of the sphere (*i.e.* points for which $\lambda = 1$) are projectors, and therefore correspond to pure states; points inside the sphere are mapped to statistical mixtures.

1.2 Measurements in Quantum Mechanics

In the standard presentation of quantum mechanics, measurable quantities correspond to self-adjoint operators $\hat{\mathcal{O}}$ acting on a Hilbert space $\mathcal{H}$. The possible outcomes of the measurement process are the eigenvalues

λ_i of the operator, while the probability of obtaining the outcome λ_i is given by $|\langle\psi|\varphi_i\rangle|^2$, with $\hat{\mathcal{O}}\varphi_i = \lambda_i\varphi_i$, given that the state of the system is $|\psi\rangle\langle\psi|$.
After the measurement the system is left in the state defined by the projection of $|\psi\rangle\langle\psi|$ on the eigenspace corresponding to the measurement outcome.
The expectation value of a measurement is then obtained as:

$$\begin{aligned}\langle\hat{O}\rangle &= \sum_{i=0}^{d}\lambda_i p_i = \sum_{i=0}^{d}\lambda_i|\langle\psi|\varphi_i\rangle|^2 \\ &= \sum_{i=0}^{d}\langle\psi|\lambda_i|\varphi_i\rangle\langle\varphi_i|\psi\rangle = \langle\psi|\hat{O}|\psi\rangle = \mathrm{tr}[\hat{O}\rho_\psi]\end{aligned} \tag{1.9}$$

where $\rho_\psi = |\psi\rangle\langle\psi|$ is the density operator corresponding to the pure state $|\psi\rangle$.

Relation (1.9) can be extended to act also on density matrices that correspond to mixed states, taking $\rho = p\,|\psi\rangle\langle\psi| + (1-p)\,|\psi'\rangle\langle\psi'|$ we find:

$$\langle\hat{O}\rangle = \mathrm{tr}[\hat{O}\rho] = p\;\;\mathrm{tr}[\hat{O}\,|\psi\rangle\langle\psi|] + (1-p)\;\;\mathrm{tr}[\hat{O}\,|\psi'\rangle\langle\psi'|] \tag{1.10}$$

which is indeed the correct expression for the expectation value of $\hat{O}$ on the mixed state.

Such kind of measurements are commonly referred to as Projection-Valued Measures (PVMs), as they can be defined by the set of orthonormal projectors $\{|\varphi_i\rangle\langle\varphi_i|\}$ that make up the corresponding observable. Each element of the set matches one of the possible outcomes of the measurement process.

1.2.1 Limitations of PVMs

PVMs, as we will see, are not the most general form of quantum measurement one can devise. Although they are usually presented as the standard formalization of measurement, there are many procedures that extract information about a quantum system that cannot be formalized by PVMs.

Example 1.1. A measurement device on a qubit acts as follows: A classical 'coin flip' experiment is used to decide the fate of the qubit. In one case, a projective measurement along the z axis will take place, and in the other case the qubit will be measured along the x axis. The outcome of the process is just the outcome of the chosen measurement, but no information about which PVM has been performed is available. The setup is summarized in figure:

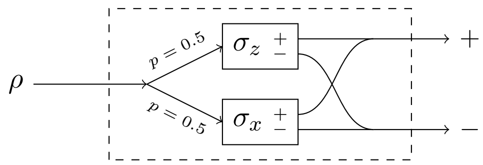


FIGURE 1.1

Defining[3] $E_+ = \frac{1}{2}(P_0 + P_+)$ and $E_- = \frac{1}{2}(P_1 + P_-)$, and calling ρ the state of the input qubit one can easily check that the outcome probabilities for this set up are given by:

$$p_\pm = \mathrm{tr}[E_\pm \rho] \tag{1.11}$$

Unfortunately $E_\pm^2 \neq E_\pm$ and therefore the set $\{E_\pm\}$ is not a PVM. A closer look at the definitions of $E_\pm$ reveals that the above procedure can be thought as a statistical mixture of the two PVMs $\{P_0, P_1\}$ and $\{P_+, P_-\}$.

[3] $P_i = |i\rangle\langle i|\,;\quad i = 0, 1, +, -.$

Example 1.2. Consider the following measurement procedure: the system being measured is entangled with another auxiliary quantum system which has been prepared in a known state, and a projective measurement is then performed on the auxiliary system. As we will see in detail in the next section, this kind of measurement can give information about the system that can not be obtained by performing a standard PVM on the initial system alone (e.g. a 3 outcome measurement on a 2 dimensional system).

1.2.2 Positive Operator Valued Measures (POVMs)

The more general description of a measurement is given by the probabilities of obtaining a certain outcome when measuring a (pure or mixed) state ρ. With this in mind, we can associate to each outcome i a function $\epsilon_i : \mathcal{H} \to [0,1]$ such that $\epsilon_i(\rho)$ is the probability of obtaining the outcome i when measuring the state ρ. We will call such functions *effects*[2].
Effects can be represented as positive operators E acting on $\mathcal{H}$ as follows: $\epsilon(\rho) = \mathrm{tr}[E\rho]$, with $\hat{0} \leq E \leq \mathbb{1}$[4]. The name effect will be used to refer both to the function ϵ and to the corresponding operator E.

Definition 1.4. A *Positive Operator Valued Measure* (POVM) is a set of effects $\{E_i\}_{i=1}^n$ such that $\sum_{i=1}^n E_i = \mathbb{1}$.

Confronting this with a standard PVM (*i.e.*, a set of projectors summing up to the identity) we can see that the only difference is the additional requirement for the elements of PVMs to be orthonormal[5]. Therefore, all PVMs are also POVMs.

[4]With the notation $\hat{0} \leq E \leq \mathbb{1}$ it is meant that $0 \leq \epsilon(\rho) \leq 1 \ \forall \rho \in S(\mathcal{H})$.
[5]Formally, for a PVM $\{P_i\}$ we require $P_i P_j = \delta_{ij} P_i$

With this new definition of measurement, we can overcome the limitations presented in the previous section:

Example 1.3. Referring to the setup presented in example 1.1, we can check that the set of operators $\{E_\pm\}$ introduced there constitute a POVM: they are positive (they are defined as sum of positive operators) and:

$$\sum_{i=\pm} E_i = \frac{1}{2}(P_0 + P_1 + P_+ + P_-) = \frac{1}{2}(\mathbb{1} + \mathbb{1}) = \mathbb{1} \tag{1.12}$$

Example 1.4. The process described in example 1.2 can be formalized as follows: the input state ρ_s is coupled to the ancillary system giving a total state $\rho_{sa} = \rho_s \otimes \rho_a$. A general interaction between the systems can be described with a unitary transformation U:

$$\rho_{sa} \to U\rho_{sa}U^\dagger \tag{1.13}$$

Performing a standard projective measurement $\{P_i\}$ on the ancilla gives the following probabilities for the outcomes:

$$p_i = \mathrm{tr}[(\mathbb{1} \otimes P_i)\ (U\rho_s \otimes \rho_a U^\dagger)] \tag{1.14}$$

Using the cyclic property of the trace and rearranging, we can rewrite:

$$p_i = \mathrm{tr}[U^\dagger(\mathbb{1} \otimes P_i)U\ (\rho_s \otimes \rho_a)] = \mathrm{tr}_S\left[\rho_s\ \mathrm{tr}_A[U^\dagger(\mathbb{1} \otimes P_i)U(\mathbb{1} \otimes \rho_A)]\right] \tag{1.15}$$

Defining now $E_i = \mathrm{tr}_A[U^\dagger(\mathbb{1} \otimes P_i)U(\mathbb{1} \otimes \rho_A)]$, we find that the outcomes probabilities can be obtained from an expression involving only states and operators acting on the input system side:

$$p_i = \mathrm{tr}[\rho_s E_i] \tag{1.16}$$

We will now show that the set $\{E_i\}$ defined as above is indeed a POVM. The positivity condition is guaranteed by p_i being probabilities (and from the fact that we made no assumption on the form of ρ_s), while

$$\sum_i E_i = \mathrm{tr}_A[U^\dagger(\mathbb{1}\otimes\sum_i P_i)U(\mathbb{1}\otimes\rho_A)] = \mathrm{tr}_A[U^\dagger U(\mathbb{1}\otimes\rho_A)] = \mathbb{1}\cdot\mathrm{tr}_A[\rho_A] = \mathbb{1} \tag{1.17}$$

These two examples show how POVMs can be used to describe measurement procedures which are not formalizable by using only PVMs.

Theorem 1.5. *A POVM $\{E_i\}_{i=1}^N$, on a d-dimensional system $\mathcal{H}$ can be realized as a PVM in an N-dimensional extension of $\mathcal{H}$, $\mathcal{H}_{ext} = \mathcal{H} \oplus \mathcal{H}_A$*

Proof. Without loss of generality we can restrict ourselves to the case where all effects are rank-1 operators, as any POVM can be decomposed as a set pure effects $\{E_i = |\psi_i\rangle\langle\psi_i|\}$, where $|\psi_i\rangle$ are subnormalized states. We will also assume $N > d$, as the case $N \leq d$ is completely trivial.
The conclusion is equivalent to finding a set of N orthogonal projectors $M = \{P_i\}$ on $\mathcal{H}_{ext}$ (*i.e.* a PVM) such that the effects composing the POVM can be obtained projecting the set M in the subspace $\mathcal{H}$ of $\mathcal{H}_{ext}$, formally:

$$\mathrm{tr}[\rho E_i] = \mathrm{tr}[\rho\Pi P_i \Pi] = \mathrm{tr}[\Pi\rho\Pi P_i] = \mathrm{tr}[\rho_{ext} P_i] \tag{1.18}$$

where Π is the operator projecting $\mathcal{H}_{ext}$ on $\mathcal{H}$.
We can chose the projectors P_i as the projectors on the states:

$$|\phi_i\rangle = |\psi_i\rangle + \sum_{s=d+1}^{N} c_{is} |\varphi_s\rangle \tag{1.19}$$

where $|\varphi_s\rangle$ are $N-d$ vectors orthogonal to each other and to all $|\phi_i\rangle$ and c_{is} are coefficients to be determined.
It is now sufficient to prove that it is always possible to choose the coefficients c_{is} in such a way that the $|\phi_i\rangle$ form an orthonormal basis. To do this, choose a basis for $\mathcal{H}$ and denote with a_{ik}, $k = 1...d$ the coefficients of the expansion of the $|\psi_i\rangle$ on this basis. Consider now the following matrix:

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,d} & c_{1,d+1} & \dots & c_{1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{N,1} & \dots & a_{N,d} & c_{N,d+1} & \dots & c_{N,N} \end{bmatrix} \tag{1.20}$$

The rows of this matrix are just the representation of the $|\phi_i\rangle$ on some basis. Since $\sum |\psi_i\rangle \langle\psi_i| = \mathbb{1}$, we have $\delta_{ij} = \sum_l^N a_{il}^* a_{lj}$, hence, the first d columns of the matrix are orthonormal. There are now infinitely many ways to choose the c_{ij} for all the columns (and therefore the rows) to be orthonormal. □

In general, this way of realizing a POVM will not have a direct physical interpretation, since not all systems' Hilbert spaces can be extended as required in the theorem. However:

Proposition 1.6. *Acting with a POVM $\{E_i\}_{i=1}^N$ on a system S is equivalent to acting with a PVM on a larger system of which S is a subsystem.*

A constructive proof of this can be found in Ref. [3]. A more general proof of this fact will be given later in the discussion.

1.2.3 Quantum Channels

Let us pause our discussion about measurements and focus on a step that usually precedes the measurement in most experimental schemes: the system, before being measured, undergoes some transformation that alter its state. In textbook quantum theory, these kind of state transformation are represented by a unitary operator U acting on the state, that is: $\rho' = U\rho U^\dagger$. The operator U is usually implemented as a time evolution of the system, driven by a carefully chosen Hamiltonian so that $U = e^{iHt}$. This kind of formalization is however, not sufficient to describe the full set of transformation a system can udergo.

Example 1.5. Consider a photon travelling in an optical fiber: due to the interaction with the fiber, the quantum state of the photon will couple with the fiber in which it is travelling, causing decoherence in the photon state. While the action on the total system (fiber + photon) is unitary, the state transformation that the photon undergoes can not be described by a unitary operation: we need a more general formal device to describe such scenarios.

The most general operator on $S(\mathcal{H})$ that can describe a state transformation has to be - in order to preserve probabilities - linear, completely positive and trace preserving. The only non obvious requirement is complete positivity: Consider the map V_A acting on a subsystem A. We can extend its action to a larger space, for instance:

$$(V_A \otimes \mathbb{1}_B)(\rho_A \otimes \rho_B) = V_A(\rho_a) \otimes \rho_B \tag{1.21}$$

The positivity of V_A is not enough to ensure the positivity of $V_A \otimes \mathbb{1}_B$, which has to be required in order to avoid negative probabilities arising in the extended space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_B$. Therefore, we require that $V_A \otimes \mathbb{1}_B$ to be positive for any extension B; this requirement is called complete positivity.

Example 1.6. An example of a positive but not completely positive operator is the *partial transposition.*
Consider a Hilbert space $\mathcal{H}$ and a basis $\{\varphi_i\}_{i=1}^N$ over it. The transpose operator can be defined on this basis as: $\tau(|\varphi_i\rangle\langle\varphi_j|) = |\varphi_j\rangle\langle\varphi_i|$. Since transposing an operator does not change its eigenvalues, τ is a positive operator.
Consider now the extended operator $\tau_A \otimes \mathbb{1}_B$ acting on the density operator $|\psi\rangle\langle\psi| = \frac{1}{N}\sum_{i,j}|\varphi_i\rangle\langle\varphi_j| \otimes |\varphi_i\rangle\langle\varphi_j|$. The result is:

$$\tau_A \otimes \mathbb{1}_B |\psi\rangle\langle\psi| = \frac{1}{N}\sum_{i,j}|\varphi_j\rangle\langle\varphi_i| \otimes |\varphi_i\rangle\langle\varphi_j| \tag{1.22}$$

This operator is not positive, as acting on $|\psi_-\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle - |\varphi_2\rangle \otimes |\varphi_1\rangle$ gives:

$$\left(\sum_{i,j}|\varphi_j\rangle\langle\varphi_i| \otimes |\varphi_i\rangle\langle\varphi_j|\right)|\psi_-\rangle = -|\psi_-\rangle \tag{1.23}$$

Therefore, we must conclude that transposition is a positive but a not completely positive operator.

Proposition 1.7. *Let $S \in \mathcal{L}(\mathcal{H}_\mathcal{A})$ be an operator. The map $N_S(T) = STS^\dagger$ is linear and completely positive.*

Proof. The linearity is manifest in the definition. Consider now a positive operator O on an extended space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. For every vector $|\psi\rangle \in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$

we have:

$$\begin{aligned} \langle\psi| (N_S \otimes \mathbb{1}_B) O |\psi\rangle &= \langle\psi| (S \otimes \mathbb{1}_B) O (S^\dagger \otimes \mathbb{1}_B) |\psi\rangle \\ &= \langle\psi_S| O |\psi_S\rangle \geq 0 \end{aligned} \tag{1.24}$$

since O is positive. Therefore $(N_S \otimes \mathbb{1}_B)$ is positive and N_S is completely positive. □

With the above considerations in mind, we are now ready to define:

Definition 1.8. A *Quantum channel* is a map $V : S(\mathcal{H}) \to S(\mathcal{H}')$ such that:

- is linear
- is completely positive
- is trace preserving[6]

We now present a few important results regarding quantum channels that will become useful later in the discussion.

Theorem 1.9. *(Stinespring's theorem) If $\tau : S(\mathcal{H}) \to S(\mathcal{H})$ is a channel, then there exists a Hilbert space $\mathcal{H}_E$, a pure state $\eta \in \mathcal{H}_E$ and a unitary U on $\mathcal{H} \otimes \mathcal{H}_E$ such that:*

$$\tau(\rho) = \mathrm{tr}_E[U \ (\rho \otimes \eta) \ U^\dagger] \quad \forall \rho \in \mathcal{H} \tag{1.25}$$

Proof. The proof can be found in [4, p. 211-216]. □

The meaning of this result is that the action of any channel on a quantum system can be physically realized by coupling it to an auxiliary system and

[6] $\mathrm{tr}[V(\rho)] = \mathrm{tr}[\rho] \ \ \forall \rho \in S(\mathcal{H})$

performing a carefully chosen time evolution on the composite system. Moreover, since equation (1.25) can be used to define a channel (given $\eta \in \mathcal{H}_E$ and U), we have that time evolutions in open quantum systems can be described using quantum channels.

Theorem 1.10. *(Kraus decomposition) A linear map $\tau : T(\mathcal{H}) \to T(\mathcal{H})$ is a channel if and only if there exists a sequence of bounded operators $A_1, A_2, A_3, ...$ such that:*

$$\tau(T) = \sum_i A_i T A_i^\dagger \qquad \sum_i A_i^\dagger A_i = \mathbb{1} \tag{1.26}$$

Proof. Proposition 1.7 ensures that each map $N_{A_k}(T) = A_k T A_k^\dagger$ is linear and completely positive (and so is τ). We also have:

$$\mathrm{tr}[\tau(T)] = \sum_i \mathrm{tr}[A_i T A_i^\dagger] = \sum_i \mathrm{tr}[A_i^\dagger A_i T] = \mathrm{tr}\left[\left(\sum_i A_i^\dagger A_i\right) T\right] = \mathrm{tr}[T] \tag{1.27}$$

therefore τ is a channel.

We will now prove the other direction. By theorem 1.9 we have that τ admits a dilation $(\mathcal{H}_E, U, \eta)$. We can always write $\eta = |\varphi_1\rangle\langle\varphi_1|$ with $|\varphi_1\rangle \in \mathcal{H}_E$; we can also choose a basis $\{\varphi_i\}_{i=1}^d$ for $\mathcal{H}_E$ and define the action of the operators A_k as:

$$\langle\psi| A_k |\tilde{\psi}\rangle = \langle\psi \otimes \varphi_k| U |\tilde{\psi} \otimes \varphi_1\rangle \quad \forall \psi, \tilde{\psi} \in \mathcal{H} \tag{1.28}$$

Since $|\langle\psi| A_k |\tilde{\psi}\rangle| = |\langle\psi \otimes \varphi_k| U |\tilde{\psi} \otimes \varphi_1\rangle| \leq |\psi|\, |\tilde{\psi}|\, |U|$ we conclude that the A_k are bounded operators.

We also have:

$$\begin{aligned}
\langle\psi|\,\tau(|\eta\rangle\langle\eta|)\,|\tilde{\psi}\rangle &= \langle\psi|\,\mathrm{tr}_E\left[U\,|\eta\rangle\langle\eta|\otimes|\varphi_1\rangle\langle\varphi_1|\,U^\dagger\right]|\tilde{\psi}\rangle \\
&= \sum_k \langle\psi\otimes\varphi_k|\,U\,|\eta\rangle\langle\eta|\otimes|\varphi_1\rangle\langle\varphi_1|\,U^\dagger\,|\tilde{\psi}\otimes\varphi_k\rangle \\
&= \sum_k \langle\psi\otimes\varphi_k|\,U\,|\eta\otimes\varphi_1\rangle\langle\eta\otimes\varphi_1|\,U^\dagger\,|\tilde{\psi}\otimes\varphi_k\rangle \\
&= \sum_k \langle\psi|\,A_k\,|\eta\rangle\langle\eta|\,A_k^\dagger\,|\tilde{\psi}\rangle
\end{aligned} \tag{1.29}$$

and therefore $\tau(|\eta\rangle\langle\eta|) = \sum_k A_k\,|\eta\rangle\langle\eta|\,A_k^\dagger$. We also have:

$$\mathrm{tr}[T] = \mathrm{tr}[\tau(T)] = \mathrm{tr}\left[\sum_k A_k^\dagger A_k T\right] \quad \forall T \in S(\mathcal{H}) \tag{1.30}$$

thus $\sum_k A_k^\dagger A_k = \mathbb{1}$.

□

Proposition 1.11. *Two finite sets $\{A_i\}_i = 1^n$ and $\{B_i\}_i = 1^m$ of bounded operators define the same channel if and only if:*

$$A_j = \sum_{k=1}^{m} u_{jk} B_k \tag{1.31}$$

where the matrix u_{ik} is unitary.

Proof. The proof of this fact can be found in [2, p. 190-191]. □

Theorem 1.12. *(Choi's theorem) Let τ be a positive linear map on $\mathcal{L}(\mathcal{H})$, with $\dim(\mathcal{H}) = d$. The following statements are equivalent:*

i) τ is completely positive

ii) $\mathbb{1}_d \otimes \tau$ *is positive*

iii) The matrix ($\{\varphi_i\}_{i=1}^d$ *is any basis for* $\mathcal{H}$*):*

$$\Phi_\tau = \frac{1}{d} \begin{bmatrix} \tau(|\varphi_1\rangle\langle\varphi_1|) & \dots & \tau(|\varphi_1\rangle\langle\varphi_d|) \\ \vdots & \ddots & \vdots \\ \tau(|\varphi_d\rangle\langle\varphi_1|) & \dots & \tau(|\varphi_d\rangle\langle\varphi_d|) \end{bmatrix} \tag{1.32}$$

is positive.

Proof. From the definition of complete positivity we have $(i) \Rightarrow (ii)$. Consider now the positive matrix

$$M = \frac{1}{d}\sum_{jk} |jj\rangle\langle kk| = |\psi_-\rangle\langle\psi_-| \quad \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \tag{1.33}$$

Noting that:

$$(\mathbb{1}_d \otimes \tau)M = \frac{1}{d}\sum_{jk} |j\rangle\langle k| \otimes \tau(|j\rangle\langle k|) = \Phi_\tau \tag{1.34}$$

and using the positivity of $(\mathbb{1}_d \otimes \tau)$ and M we have the positivity of Φ_τ and therefore $(ii) \Rightarrow (iii)$.

We will now prove $(iii) \Rightarrow (i)$. Denote with $\phi_l \in \mathbb{C}^d \otimes \mathbb{C}^d \quad l = 1...n$ the unnomrmalized eigenvectors of the positive operator Φ_τ, so that $\Phi_\tau = \sum_l^n |\phi_l\rangle\langle\phi_l|$. They are orthogonal and linearly independent. Looking at $\mathbb{C}^d \otimes \mathbb{C}^d$ as the direct sum $\mathbb{C}^d \oplus \cdots \oplus \mathbb{C}^d$, we can define P_j as the projector on the jth 'copy' of $\mathbb{C}^d$. We now have:

$$\tau[|\varphi_j\rangle\langle\varphi_k|] = P_j \Phi_\tau P_k = \sum_l P_j |\phi_l\rangle\langle\phi_l| P_k \tag{1.35}$$

Define now n operators by the relation $V_l\,|\varphi_j\rangle = P_j\,|\phi_l\rangle$. Thus:

$$\tau[|\varphi_j\rangle\langle\varphi_k|] = \sum_l V_l\,|\varphi_j\rangle\langle\varphi_k|\,V_l^\dagger \tag{1.36}$$

We now have $\tau(T) = \sum_l V_l T V_l^\dagger \quad \forall T \in \mathcal{L}(\mathcal{H})$, using Proposition 1.7 we conclude the proof. □

Definition 1.13. Given a completely positive linear map $\tau : S(\mathcal{H}) \to S(\mathcal{H})$, we define the Heisenberg picture of τ as a map τ^H such that:

$$\mathrm{tr}[\tau^H(T)\rho] = \mathrm{tr}[T\tau(\rho)] \quad \forall \rho \in S(\mathcal{H}), T \in \mathcal{T}(\mathcal{H})^7 \tag{1.37}$$

Proposition 1.14. *Given a channel $\tau : S(\mathcal{H}) \to S(\mathcal{H})$, its Heisenberg picture representation τ^H can be obtained as:*

$$\tau^H(T) = \mathrm{tr}_E[V^\dagger(T \otimes \mathbb{1})V] \tag{1.38}$$

Where $V = U(\mathbb{1} \otimes \sqrt{\eta})$ is an isometry on $\mathcal{H}_E$, with U, η and $\mathcal{H}_E$ as in theorem 1.9.

Proof. From the definition of τ^H we have:

$$\mathrm{tr}[\tau^H(T)\rho] = \mathrm{tr}[T\tau(\rho)] \quad \forall \rho \in S(\mathcal{H}), T \in [0, \mathbb{1}] \tag{1.39}$$

[7] $\mathcal{T}(\mathcal{H}) = \{T is linear and\, \mathrm{tr}[\sqrt{T^\dagger T}] < +\infty\}$

From theorem 1.9 we have:

$$\begin{aligned}
\mathrm{tr}[\tau^H(T)\rho] &= \mathrm{tr}\left[T\ \ \mathrm{tr}_E[U(\rho\otimes\eta)U^\dagger]\right] = \mathrm{tr}\left[(T\otimes\mathbb{1})\ (U(\rho\otimes\eta)U^\dagger)\right] \\
&= \mathrm{tr}\left[U^\dagger(T\otimes\mathbb{1})U(\rho\otimes\eta)\right] = \mathrm{tr}\left[\mathrm{tr}_E[U^\dagger(T\otimes\mathbb{1})U(\mathbb{1}\otimes\eta)]\ \rho\right] \\
&= \mathrm{tr}\left[\mathrm{tr}_E[V^\dagger(T\otimes\mathbb{1})V]\rho\right] \qquad \forall\rho\in S(\mathcal{H}), T\in[0,\mathbb{1}]
\end{aligned} \tag{1.40}$$

concluding the proof. □

1.2.4 Measurement models

When performing a measurement on a system, there are usually many ways to devise a physical procedure that implements the measurement and yields the correct outcome probabilities.

A typical way of realizing a measurement has been described in Examples 1.2 and 1.4, with the system under study being coupled with an auxiliary system -called *probe system*- and by then using a POVM -called *pointer observable*- on the probe. A *measurement model* is a formal description of such a procedure:

Definition 1.15. A *measurement model* is a quadruple $(\mathcal{K}, \chi, V, \{F_i\})$ such that:

- $\mathcal{K}$ is the Hilbert space of the probe system.
- χ is the initial state of the probe
- V is a channel from $\mathcal{H}\otimes\mathcal{K}$ to itself, describing the coupling between the system and the probe.

- $\{F_i\}$ is a POVM on the probe (the pointer observable).

Definition 1.16. Given a POVM $\{E_i\}$ on a system, we say that $M = (\mathcal{K}, \chi, V, \{F_i\})$ is a measurement model for $\{E_i\}$ if:

$$\mathrm{tr}[\rho E_i] = \mathrm{tr}[V(\rho \otimes \chi)V^\dagger \ (\mathbb{1} \otimes F_i)] \quad \forall \rho \in S(\mathcal{H}) \tag{1.41}$$

Remark 1.17. Note that the behaviour of the channel V need not to be specified full; since the probe is always prepared in the state χ, it's only necessary to specify the action of the channel on the subspace $T(\mathcal{H}) \otimes range(|\chi\rangle \langle\chi|)$.

1.2.5 Instruments

Until now we focused our discussion on the description of the outcome probabilities for some measurement procedure. We now want to shift our attention to the state of the measured system after the measurement has been performed.

In order to derive the after measurement state, we suppose that, after measuring $\{E_i\}$ on the state ρ by means of some measurement model $M = (\mathcal{K}, \chi, V, \{F_i\})$, we measure the observable $\{B_j\}$ on the system. In this scenario, the probability of getting the outcome i for the first measurement and j for the second can be expressed as:

$$\begin{aligned} p(i,j) &= \mathrm{tr}[V(\rho \otimes \chi)V^\dagger \ (B_j \otimes F_i)] \\ &= \mathrm{tr}[B_j \ \mathrm{tr}_{\mathcal{K}}[V(\rho \otimes \chi)V^\dagger \ (\mathbb{1} \otimes F_i)]] \end{aligned} \tag{1.42}$$

On the other hand, if we denote with $\tilde{\rho}_i^M$ the post-measurement state we get by obtaining outcome i when measuring M, the same probability can be written as:

$$p(i,j) = p(j|i)\ p(i) = \mathrm{tr}[B_j \tilde{\rho}_i^M]\ p(i) \tag{1.43}$$

Defining the unnormalized state $\rho_i^M = \tilde{\rho}_i^M\ p(i)$, by confronting (1.42) and (1.43) we obtain:

$$\rho_i^M = \mathrm{tr}_{\mathcal{K}}[V(\rho \otimes \chi)V^\dagger\ (\mathbb{1} \otimes F_i)] \tag{1.44}$$

For each outcome i, equation (1.44) defines a linear mapping $\mathcal{I}_i^M(\rho) = \rho_i^M$ that maps pre-measurement states into (unnormalized) post-measurement states. Note that $\mathrm{tr}[\mathcal{I}_i^M(\rho)] = \mathrm{tr}[\rho_i^M] = p(i)$.

Definition 1.18. An *instrument* is a set of maps $\{\mathcal{I}_i\}$ such that:

- $\mathcal{I}_i$ is linear and completely positive
- $\mathrm{tr}[\sum_i \mathcal{I}_i(\rho)] = 1 \quad \forall \rho \in T(\mathcal{H})$
- $\mathrm{tr}[\mathcal{I}_{\dot{\cup}_{i\in Y}}(\rho)] = \sum_{i\in Y} \mathrm{tr}[\mathcal{I}_i(\rho)]$, where Y is a set of possible outcomes.

Remark 1.19. Note that, given an instrument $\{\mathcal{I}_i\}$, the sum of it's elements $\sum_i \mathcal{I}_i$ defines a channel.

Definition 1.20. Given a POVM $\{E_i\}$ on a system and and an instrument $\{\mathcal{I}_i\}$, we say that the instrument is *E-compatible* if:

$$\mathrm{tr}[\mathcal{I}_i(\rho)] = \mathrm{tr}[E_i \rho] \tag{1.45}$$

That is, if the instrument reproduces the outcome probabilities of the meausrement described by the POVM.

Proposition 1.21. *Given the POVM $\{E_i\}$, we will show that the set $\{\mathcal{I}_i^E\}$, where:*

$$\mathcal{I}_i^E(\rho) = \sqrt{E_i}\rho\sqrt{E_i} \tag{1.46}$$

is an E-compatible instrument.

Proof. For each i, the map defined in eq. (1.21) is linear and completely positive (Proposition 1.7). Furthermore, using the cyclic property of the trace and the fact that $\sum_i E_i = \mathbb{1}$, we easily get that $\mathrm{tr}[\sum_i \mathcal{I}_i(\rho)] = 1$. We also have - for disjoint outcomes i and j - that $E_{i\cup j} = E_i + E_j$, therefore, using again the cyclic property of the trace we prove the last condition of definition 1.18 and that $\{\mathcal{I}_i^E\}$ is indeed an instrument. Its E-compatibility follows again from the properties of the trace. □

Theorem 1.22. *(Ozawa's Theorem) For every instrument $\{\mathcal{I}_i\}$ there is a measurement model $M = (\mathcal{K}, \chi, V, \{F_i\})$ such that*

$$\mathrm{tr}_{\mathcal{K}}[V(\rho \otimes \chi)V^\dagger \ (\mathbb{1} \otimes F_i)] = \mathcal{I}_i^M(\rho) = \mathcal{I}_i(\rho) \quad \forall \rho \in S(\mathcal{H}) \tag{1.47}$$

where V is a unitary and $\{F_i\}$ is a PVM.

Proof. Each element of the Instrument I_i admits a Kraus decomposition $\mathcal{I}_i(\rho) = \sum_j A_j \rho A_j^\dagger$. The sum of all elements in the instrument gives a decomposition for the channel $I_\Omega = \sum_i \mathcal{I}_i$; this decomposition contains a finite number, say N, of operators.
Using theorem 1.9 we find a dilation $(\mathcal{H}_E, \eta, U)$ such that $\mathcal{I}_\Omega(\rho) = \mathrm{tr}_E[U(\rho \otimes |\eta\rangle\langle\eta|)U^\dagger]$. Without loss of generality, the dimension of $\mathcal{H}_E$ can be chosen to be equal to N, since we can always increase its size by a tensor product with another Hilbert space. Choosing an orthonormal basis $\{\varphi_i\}$ for $\mathcal{H}_E$, we can define N operators on $\mathcal{H}$ by the relation:

$$\langle\psi| B_j |\tilde{\psi}\rangle = \langle\psi \otimes \varphi_j| U |\tilde{\psi} \otimes \eta\rangle \quad \forall \psi, \tilde{\psi} \in \mathcal{H} \tag{1.48}$$

With this definition for B_j, we have $\mathcal{I}_\Omega(\rho) = \sum_j B_j \rho B_j^\dagger$. We have therefore that the two sets $\{B_j\}$ and $\{A_i\}$ are decomposition of the same channel, and by proposition 1.11 they are connected by a unitary u_{ik}.
Defining another basis for $\mathcal{H}_E$ as $|\tilde{\varphi}_i\rangle = \sum_k u_{ik} |\varphi_k\rangle$, we determine a PVM $\{\tilde{F}_i = |\tilde{\varphi}_i\rangle \langle\tilde{\varphi}_i|\}$. We now have that

$$A_j \rho A_j^\dagger = \mathrm{tr}_E[U(\rho \otimes |\eta\rangle \langle\eta|)U^\dagger(\mathbb{1} \otimes \tilde{F}_j)] \quad \forall \rho \in S(\mathcal{H}) \tag{1.49}$$

The measurement model M is then composed by the Hilbert space $\mathcal{K} = \mathcal{H}_E$, the state $\chi = \eta$, the channel defined by U through proposition 1.7, and the PVM $\{F_i\}$ obtained by opportunely grouping the outcomes of $\{\tilde{F}_i\}$. □

We are now ready to give a proof for proposition 1.6:

Proof. For any POVM $\{E_i\}$, using proposition 1.21 we can find an E-compatible instrument $\{\mathcal{I}_i^E\}$. Using Then theorem 1.22 we can find a measurement model that realizes the instrument (and therefore the POVM) by means of a PVM on a auxiliary system. □

Remark 1.23. In our discussion, we have introduced three different ways to describe a measurement on a quantum system: POVMs, measurement models and instruments. These three descriptions can be organized in a hierarchy.
POVMs characterize only the outcome probabilities of a measurement procedure, but they don't give any information on the post measurement state or on the physical procedure used to perform the observation. For each POVM, we can find many different instruments that produce its given outcome probability distribution with distinct post measurement states.

Similarly, a general instrument can by physically realized by means of different measurement models; for example, given a particular measurement model for an instrument, one could construct another by increasing the dimension of the probe system and extending the interaction channel to the 'extra' dimensions with the identity channel.

Chapter 2

Compatibility of Measurements

In this chapter we explore compatibility of quantum measurements, extending the textbook definition (*i.e.* commutativity) to the generalized measurements introduced in Chapter 1.

2.1 The usual notion of observable compatibility

In the standard formulation of quantum theory, two observables (PVMs) are said to be compatible if they are *jointly measurable*, *i.e.* if there is a procedure that can measure both observables on a system without having the outcome of one spoil the measurement of the other. In general this does not happen: after a measurement the system is left in an eigenstate of the measured observable, and therefore a generic subsequent measurement would be affected by the outcome of the first[1]. However, if it is possible to find a basis of common eigenstates for the two observables, we can determine the value of both on the same state. This is because the post measurement state will lie completely in one of the eigenspaces of each observable.

Theorem 2.1. *Two observables A and B admit a decomposition on a common basis of eigenstates $\{|\varphi_i\rangle\}$ if and only if $[A, B] = 0$.*

Proof. Denoting with μ_i and λ_i the eigenvalues of A and B respectively, it's easy to prove that commutativity follows form the existence of $\{|\varphi_i\rangle\}$:

$$\begin{aligned} AB\,|\psi\rangle &= A\sum_i \lambda_i c_i\,|\varphi_i\rangle = \sum_i \lambda_i \mu_i c_i\,|\varphi_i\rangle \\ &= B\sum_i \mu_i c_i\,|\varphi_i\rangle = BA\,|\psi\rangle \end{aligned} \tag{2.1}$$

To prove the other direction we note that, if $A\,|v\rangle = \mu_i\,|v\rangle$

$$AB\,|v\rangle = BA\,|v\rangle = B\mu_i\,|v\rangle = \mu_i B\,|v\rangle \tag{2.2}$$

And therefore that B preserves the eigenspaces of A. Now, restricting B to the eigenspace of each eigenvalue μ_i and diagonalizing B in each restriction, we find a basis of eigenstates of B wich are also eigenstates of A. □

With the above result in mind, we conclude that two PVMs are compatible (or jointly measurable) if their corresponding observables commute. A a set of PVMs is said to be compatible if each pair is.

[1]Think about position and momentum: the measurement of one destroys all information about the probability distribution of the other

2.2 Compatibility for generalized measurements

2.2.1 POVMs

We now wish to find a notion of compatibility applicable to POVMs. We will again use the idea of joint measurability.

Definition 2.2. A set of POVMs $\{E_{i|\lambda}\}$[2], is said to be *jointly measurable* or *compatible* if there is a POVM $\{G_k\}$ from which the effects $E_{i|\lambda}$ can be attained as:

$$E_{i|\lambda} = \sum_k p_k(i|\lambda) G_k \tag{2.3}$$

where $p_k(i|\lambda)$ are positive constants with $\sum_i p_k(i|\lambda) = 1$.

In practice, this means that the set $\{E_{i|\lambda}\}$ is jointly measurable if we can obtain the outcome probabilites for each POVM in the set by post-processing the outcome distribution of $\{G_k\}$.

Proposition 2.3. *A set of POVMs* $\{E_{i_1|1}\}...\{E_{i_N|N}\}$ *is jointly measurable if and only if there exist a marginal POVM* $\{G_{i_1...i_N}\}$*, such that:*

$$E_{i_\lambda|\lambda} = \sum_{i_1...i_{\lambda-1}, i_{\lambda+1}...i_N} G_{i_1...i_N} \tag{2.4}$$

Proof. The 'if' implication is trivial, as equation (2.4) is a particular type of post-processing, as required in the definition of joint measurability. We will prove the other implication for a set of two POVM, the generalization to any finite number is then straightforward.

[2] λ indexes the POVMs in the set, i the effects for each POVM. That is $\sum_i E_{i|\lambda} = \mathbb{1} \;\; \forall \lambda$

Consider the jointly measurable generalized measurements $\{E_{i|1}\}$ and $\{E_{j|2}\}$. From the definition of joint measurability we have that there exists a POVM $\{\tilde{G}_k\}$ such that:

$$E_{i|1} = \sum_k \tilde{p}_k(i|1)\tilde{G}_k \qquad E_{i|2} = \sum_k \tilde{p}_k(i|2)\tilde{G}_k \tag{2.5}$$

Define now $p_k(i,j) = \tilde{p}_k(i|1) \cdot \tilde{p}_k(j|2)$. It's easy to see that $0 < p_k(i,j) < 1$ and $\sum_{i,j} p_k(i,j) = 1$.
Choosing now:

$$G_{i,j} = \sum_k p_k(i,j)\tilde{G}_k \tag{2.6}$$

completes the proof. □

Example 2.1. Consider the generalized measurements S_x and S_z, defined by the effects $S_{\pm|x} = \frac{1}{2}(\mathbb{1} \pm \frac{1}{\sqrt{2}}\sigma_x)$ and $S_{\pm|z} = \frac{1}{2}(\mathbb{1} \pm \frac{1}{\sqrt{2}}\sigma_z)$. They represent a 'smeared' spin measurement (they can be regarded as a mixture of an actual measurement and a coin flip) from which we still obtain some information about the spin state of the system, but not all the available information is extracted.
If we consider the generalized measurement:

$$G_{i,j} = \frac{1}{4}(\mathbb{1} + \frac{i}{\sqrt{2}}\sigma_x + \frac{j}{\sqrt{2}}\sigma_z) \quad i,j \in \{-1,+1\} \tag{2.7}$$

it is easy to see that we can recover the original effects as $S_{\pm|x} = \sum_i G_{i,\pm}$ and $S_{\pm|z} = \sum_j G_{\pm,j}$.

Remark 2.4. Notice that the effects $S_{\pm|x}$ and $S_{\pm|z}$ introduced in Example 2.1 do not commute with each other, but the corresponding POVMs statisfy the condition to be jointly measurable. We therefore conclude that the commutativity criterion for joint measurability introduced for PVMs does not hold for generalized measurements.

Example 2.2. Consider now two general spin measurements defined by $\{E_+ = E, \quad E_- = \mathbb{1} - E\}$ and $\{F_+ = F, \quad F_- = \mathbb{1} - F\}$, with:

$$E = \frac{1}{2}(\mathbb{1} + \lambda\vec{n} \cdot \vec{\sigma}) \qquad F = \frac{1}{2}(\mathbb{1} + \lambda\vec{m} \cdot \vec{\sigma}) \tag{2.8}$$

We see that E and F are positive - and therefore define valid POVMs - for $0 < \lambda < 1$. We want to find for which values of the parameter λ the joint measurability of the two POVMs is spoiled.
If we assume that the two measurements are compatible, by proposition 2.3 we have that there exists a four outcome measurement $\{G_{++}, G_{+-}, G_{-+}, G_{--}\}$ such that $E = G_{++} + G_{+-}$ and $F = G_{++} + G_{-+}$. Denoting G_{++} as G, from the positivity of effects we find the following relations:

$$\begin{aligned} 0 \leq& G \\ 0 \leq& G_{+-} = E - G \\ 0 \leq& G_{-+} = F - G \end{aligned} \tag{2.9}$$

And lastly:

$$\begin{aligned} E + F &= G + G_{+-} + G_{-+} + G_{--} \\ &= \mathbb{1} - G_{--} + G \\ &\leq \mathbb{1} + G \end{aligned} \tag{2.10}$$

Without loss of generality we can parametrize G as $G = \frac{1}{2}(\gamma\mathbb{1} + \vec{g}\cdot\vec{\sigma})$. With this parametrization the inequalities (2.9) and (2.10) become:

$$\|\vec{g}\| \leq \gamma \tag{2.11}$$

$$\|\lambda\vec{n} - \vec{g}\| \leq 1-\gamma \tag{2.12}$$

$$\|\lambda\vec{m} - \vec{g}\| \leq 1-\gamma \tag{2.13}$$

$$\|\lambda\vec{n} + \lambda\vec{m} - \vec{g}\| \leq \gamma \tag{2.14}$$

Using the triangle inequality for pairs (2.11), (2.14) and (2.13), (2.12) we get:

$$\|\lambda\vec{n} + \lambda\vec{m}\| \leq 2\gamma \qquad\qquad \|\lambda\vec{n} - \lambda\vec{m}\| \leq 2-2\gamma \tag{2.15}$$

Adding them together gives a necessary condition for the two POVMs to be jointly measurable:

$$\lambda\|\vec{n} + \vec{m}\| + \lambda\|\vec{n} - \vec{m}\| \leq 2 \tag{2.16}$$

By taking $\vec{g} = \frac{1}{2}(\lambda\vec{n} + \lambda\vec{m})$ and $\gamma = \|\vec{g}\|$ we see that this condition is also sufficient.

We conclude that the two generalized measurement are compatible if and only if:

$$\lambda \leq \frac{2}{\|\vec{n} + \vec{m}\| + \|\vec{n} - \vec{m}\|} = \frac{1}{\sqrt{1+\sin(\theta)}} \tag{2.17}$$

Where $0 < \theta < \pi$ is the angle between $\vec{n}$ and $\vec{m}$. For orthogonal spin measurement, the requirement becomes $\lambda \leq \frac{1}{\sqrt{2}}$.

2.2.2 Instruments

Following the same path we adopted for POVMs, we will call two instrument compatible if they satisfy the following criterion:

Definition 2.5. A set of instruments $\{\mathcal{I}_{i|\lambda}\}$[3], is said to be *jointly measurable* or *compatible* if there exists a general instrument $\{\mathcal{G}_k\}$ from which the maps $\mathcal{I}_{i|\lambda}$ can be attained as:

$$\mathcal{I}_{i|\lambda} = \sum_k p_k(i|\lambda)\mathcal{G}_k \tag{2.18}$$

Where $p_k(i|\lambda)$ are positive constants with $\sum_i p_k(i|\lambda) = 1$.

However, for instruments it's possible to give another - weaker - definition of compatibility:

Definition 2.6. A set of instruments $\{\mathcal{I}_{i|\lambda}\}$ is said to be *weakly compatible* if the sum of the elements for each instrument gives the same channel. That is:

$$\sum_i \mathcal{I}_{i|\lambda} = \Lambda \quad \forall\lambda \tag{2.19}$$

for some channel Λ.

Proposition 2.7. *Every set of compatible instruments $\{\mathcal{I}_{i|\lambda}\}$ is also weakly compatible.*

Proof.

$$\sum_i \mathcal{I}_{i|\lambda} = \sum_{i,k} p_k(i|\lambda)\mathcal{G}_k = \sum_k \mathcal{G}_k = \Lambda \tag{2.20}$$

[3]As before, λ indexes the instruments in the set, i the elements in each instrument. That is $\mathrm{tr}[\sum_i \mathcal{I}_{i|\lambda}(\rho)] = 1 \quad \forall\ \lambda, \rho$

where we used the fact that $\sum_i p_k(i|\lambda) = 1$ when summing over i. The final result does not depend on λ, and therefore all instruments of a compatible set sum up to the same channel. □

Definition 2.8. An instrument $\{\mathcal{I}_i\}$ is *compatible* with a channel Λ if $\sum_i \mathcal{I}_i = \Lambda$.

Proposition 2.9. *Let Λ be a channel on $\mathcal{H}$ with a minimal*[4] *dilation (see proposition 1.14):*

$$\Lambda^H(T) = \mathrm{tr}_E[V^\dagger(T \otimes \mathbb{1})V] \tag{2.21}$$

An instrument $\{\mathcal{I}_i^H\}$ [5] *is compatible with the channel Λ^H if and only if there is is a POVM $\{A_i\}$ such that:*

$$\mathcal{I}_i^H(T) = \mathrm{tr}_E[V^\dagger(T \otimes A_i)V] \tag{2.22}$$

If such POVM exists, then it is unique.

Proof. A proof of this fact can be found in [5]. □

Corollary 2.10. *Given an instrument $\{\mathcal{I}_i^H\}$ in the Heisenberg picture, as in eq (2.22), the Schrödinger picture can be written as:*

$$\mathcal{I}_i(\rho) = \mathrm{tr}_E[(\mathbb{1} \otimes A_i)V(\rho \otimes \mathbb{1})V^\dagger] \qquad \forall \rho \in S(\mathcal{H}), T \in \mathcal{T}(\mathcal{H}) \tag{2.23}$$

[4] Minimal dilations are realized taking the ancillary space $\mathcal{H}_E$ to have minimal dimension

[5] We are using the Heisenberg representation of the instrument, as defined in definition 1.13

Proof.

$$\begin{aligned}
\mathrm{tr}[T\ \mathcal{I}_i(\rho)] &= \mathrm{tr}[\mathcal{I}_i^H(T)\rho] = \mathrm{tr}\left[\mathrm{tr}_E[V^\dagger(T\otimes A_i)V]\ \rho\right] \\
&= \mathrm{tr}\left[V^\dagger(T\otimes A_i)V\ (\rho\otimes\mathbb{1})\right] \\
&= \mathrm{tr}\left[(T\otimes A_i)\ V(\rho\otimes\mathbb{1})V^\dagger\right] \\
&= \mathrm{tr}\left[(T\otimes\mathbb{1})\ (\mathbb{1}\otimes A_i)V(\rho\otimes\mathbb{1})V^\dagger)\right] \\
&= \mathrm{tr}\left[T\,\mathrm{tr}_E[(\mathbb{1}\otimes A_i)V(\rho\otimes\mathbb{1})V^\dagger]\right] \qquad \forall\rho\in S(\mathcal{H}), T\in\mathcal{T}(\mathcal{H})
\end{aligned} \tag{2.24}$$

□

Chapter 3

Steering

3.1 State steering

The concept of state steering was introduced by Schrödinger as a response to the celebrated Einstein-Pdolsky-Rosen (EPR) paper. In an EPR setup, two parties - usually denoted by Alice and Bob - share a couple of entangled particles. Steering refers to the ability of one party, say Alice, to remotely affect the other party's - say Bob's - state by performing local measurements on her side and sending the measurement settings and outcome to Bob. This reflects the fact that, for some shared states, the outcome probabilities of Bob's measurements cannot be explained by considering only classical correlations between the marginal states possessed by Bob and Alice (a practical illustration of a steerable scenario will be given in example 3.1).

Definition 3.1. Labeling with $\{A_{i|\lambda}\}$ and $\{B_{j|\mu}\}$ the generalized measurements on Alice's and Bob's sides in a EPR-like setup, where the state ρ_{AB} is shared between the two parties, we say that the scenario is *non-steerable* if the probabilities of measurement outcomes can be written as:

$$P(i,j|\lambda,\mu) = \mathrm{tr}[\rho_{AB}(A_{i|\lambda} \otimes B_{j|\mu})] = \sum_k \tilde{p}(i|\lambda,k)\ p(k)\ \ \mathrm{tr}[\rho_k B_{j|\mu}] \tag{3.1}$$

where p and $\tilde{p}$ are probability distributions and $\rho_k \in S(\mathcal{H}_B)\ \ \forall k$.

If equation 3.1 holds, Bob can just assume to have the marginal states ρ_k with probability $p(k)$. The overall probability distribution $P(i,j|\lambda,\mu)$ is then obtained considering classical correlations between Alice's and Bob's sides, which are encoded in $\tilde{p}(i|\lambda,k)$ (*i.e.* the probability for Alice to obtain outcome i given she measures POVM λ when Bob has state k). Therefore, Alice's choice of measurement cannot remotely influence Bob's state.

Example 3.1. *(Steerable scenario)* Let us assume that the two parties share the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Depending on Alice choice of measurement, say σ_x or σ_z, Bob's will find his state to be either $|x^{\pm}\rangle$ or $|z^{\pm}\rangle$. Bob has access to Alice's measurement choices and results, so he can verify that his marginal state is always equal to Alice's, and conclude that Alice measurement choice can remotely prepare his state. We say that Alice can *steer* Bob. A more detailed discussion of this scenario is given in example 4.3.

Proposition 3.2. *If Alice and Bob share a separable state, the scenario is unsteerable for any steering attempt from Alice.*

Proof. Every separable state can be written as: $\rho = \sum_k p_k \ (\rho_k^A \otimes \rho_k^B)$. Labeling Alice's and Bob's measurements as in definition 3.1 we have:

$$\begin{aligned}
P(i,j|\lambda,\mu) &= \mathrm{tr}[\rho \ (A_{i|\lambda} \otimes B_{j|\mu})] \\
&= \sum_k p_k \ \mathrm{tr}\left[(\rho_k^A \otimes \rho_k^B) \ (A_{i|\lambda} \otimes B_{j|\mu})\right] \\
&= \sum_k p_k \ \mathrm{tr}\left[\rho_k^A A_{i|\lambda}\right] \mathrm{tr}\left[\rho_k^B B_{j|\mu}\right] \\
&=: \sum_k p_k \ p(i|\lambda,k) \ \mathrm{tr}\left[\rho_k^B B_{j|\mu}\right]
\end{aligned} \tag{3.2}$$

which is the required expression for non-steerablity. □

Example 3.2. *(Werner state)* Consider that Alice and Bob share the state $W_\lambda = \lambda \left|\psi_-\right\rangle \left\langle\psi_-\right| + \frac{1-\lambda}{4}\mathbb{1}$, where $\left|\psi_-\right\rangle = \frac{1}{\sqrt{2}} \left|01\right\rangle - \left|10\right\rangle$. It is well known [6] that this state is separable for $\lambda < \frac{1}{3}$ and entangled otherwise. Somewhat surprisingly, for $\lambda < \frac{1}{2}$ we have that W_λ gives rise to local statistics for arbitrary projective measurements. This means that for $\frac{1}{3} < \lambda < \frac{1}{2}$, the state, albeit being entangled, can not be used for quantum steering.

3.1.1 State assemblages

A steering scenario is defined by both the bipartite state shared between the parties and the set of measurement Alice can choose from; the notion of state assemblage is useful to describe steering scenarios, as it contains information about both.

Definition 3.3. A *state assemblage* for a steering scenario is a set of positive matrices $\{\rho_{a|\lambda}\}$ (λ labels measurement chosen by Alice and a the outcome) such that:

$$\sum_a \rho_{a|\lambda} = \rho \quad \forall\lambda \tag{3.3}$$

where ρ is a unit-trace density matrix.

State assemblages can be used to describe the collection of unormalized partial states possesed by Bob after Alice's measurement. The requirement (3.3) is necessary to avoid signaling, *i.e* Bob must not be able to determine which measurement Alice performed.
The elements of the assemblage are obtained as[1] [2].

$$\rho_{a|\lambda} = \mathrm{tr}_A \left[\sqrt{A_{a|\lambda} \otimes \mathbb{1}}\ \rho_{AB}\ \sqrt{A_{a|\lambda} \otimes \mathbb{1}} \right] \tag{3.4}$$

Definition 3.4. A state assemblage is *unsteerable* if its elements $\rho_{a|\lambda}$ can be written as:

$$\rho_{a|\lambda} = \sum_k p_k(a|\lambda)\rho_k \tag{3.5}$$

where $\mathrm{tr}\left[\sum_k \rho_k\right] = 1$ and the $p_k(a|\lambda)$ are positive constants such that $\sum_a p_k(a|\lambda) = 1$.

3.1.2 One-way steering

In the previous discussion, we saw that a steering scenario is intrinsically asymmetrical: one of the two parties is untrusted and tries to demonstrate steering capabilities to the trusted party. Given this fundamental asymmetry, a natural question arises: are there scenarios which are ony

[1] ρ_{AB} denotes the bipartite sate shared by the parties, $\{A_{a|\lambda}\}$ is the set of measurements Alice can choose from

[2] The (unnormalized) state shared after Alice measurement is calculated using the so called Lüders instruments introduced in proposition 1.21

steerable in one direction? That is, are there scenarios in which Alice can steer Bob, yet Bob cannot steer Alice?

Example 3.3. Consider the bipartite state[3] [4]:

$$\rho_{AB} = \alpha P_{\psi_-} + \frac{1-\alpha}{5}\left(2\left|0\right\rangle\left\langle 0\right| \otimes \frac{1}{2}\mathbb{1} + \frac{3}{2}\mathbb{1} \otimes \left|1\right\rangle\left\langle 1\right|\right) \tag{3.6}$$

Similarly to a Werner state, for $\alpha < \frac{1}{2}$ a LHS model can be constructed for steering attempts from Bob with PVMs. Conversely, it is known that the scenario allows steering from Alice to Bob for $\alpha \gtrsim 0.4983$ with specifically choosen measurements. Here we omit giving the measurements and the LHS model explicitly, referring the reader to [7]. Hence the state in (3.6) is one way steerable for $0.4983 \lesssim \alpha \leq \frac{1}{2}$.

As shown in example 3.3 the answer to the aforementioned question is, perhaps surprisingly, yes.

3.2 Channel steering

In order to introduce the notion of channel steering we need to introduce first *broadcast channels*, *i.e.* channels with multiple outputs.

Definition 3.5. A *broadcast channel* $\Lambda^{C\to AB}$ is a channel $\Lambda : S(\mathcal{H}_C) \to S(\mathcal{H}_A \otimes \mathcal{H}_B)$ whose output space is composed by multiple systems.

[3] P_{ψ_-} is the projector on the singlet state $\left|\psi_-\right\rangle = \frac{1}{\sqrt{2}}(\left|01\right\rangle - \left|10\right\rangle)$

[4] $0 \leq \alpha \leq 1$

Definition 3.6. Given a broadcast channel $\Lambda^{C\to AB}$, we say that the channel $\Gamma^{C\to B}$ is a *reduction* of Λ if

$$\Gamma^{C\to B} = \mathrm{tr}_A \circ \Lambda^{C\to AB} \tag{3.7}$$

Moreover, if (3.7) holds, we say that $\Lambda^{C\to AB}$ is an *extension* of $\Gamma^{C\to B}$.

With these definition in mind, we wish to identify the steerability of the broadcast channel $\Lambda^{C\to AB}$ as the possibility for Alice to steer Bob by performing measurements on her output of the channel. In such a scenario the information available to Alice is more than just classical correlations between Alice's and Bob's outputs.

Definition 3.7. We say that $\Lambda^{C\to AB}$ is an *incoherent extension* of $\Lambda^{C\to B}$ if there is an $\Lambda^{C\to B}$-compatible instrument $\{\Lambda_i^{C\to B}\}$ and a set of states $\{\rho_i\}$ such that:

$$\Lambda^{C\to AB}(\cdot) = \sum_i \Lambda_i^{C\to B}(\cdot) \otimes \rho_i \tag{3.8}$$

We say that the extension $\Lambda^{C\to AB}$ is coherent if it is not incoherent.

Note that for an incoherent extension the information available to Alice is at most classical information about which subchannel $\Lambda_i^{C\to B}$ was applied to the input. In fact, any input ρ^C will give a separable state ρ^{AB} as output:

$$\rho^{AB} = \Lambda^{C\to AB}(\rho^C) = \sum_i \Lambda_i^{C\to B}(\rho^C) \otimes \rho_i = \sum_i p_i \rho_i^B \otimes \rho_i^A \tag{3.9}$$

Steering attempts by Alice will be made with the use of measurements;

each measurement $\{A_i\}$ on A leads to a reduction $\Lambda^{C\to B}$ of the original channel through the relation:

$$\Lambda_i^{C\to B}(\cdot) = \mathrm{tr}_A\left[(A_i \otimes \mathbb{1})\Lambda^{C\to AB}(\cdot)\right] \tag{3.10}$$

Where $\{\Lambda_i^{C\to B}\}$ is an instrument compatible with the channel $\Lambda^{C\to B}$.
We are now ready to give the definition of channel steering, the reason for such a definition will be made clear by Theorem 3.9.

Definition 3.8. We say that the channel extension $\Lambda^{C\to AB}$ is *unsteerable* if any set of POVMs $\{A_{i|\lambda}\}$ on A leads to a compatible set of instruments $\{\Lambda_{i|\lambda}^{C\to B}\}$ via the relation (3.10).

Theorem 3.9. *Every incoherent channel extension* $\Lambda^{C\to AB}$ *is unsteerable. Conversely, every compatible set of subchannels (instruments)* $\{\Lambda_{i|\lambda}^{C\to B}\}$ *for* $\Lambda^{C\to B}$ *can be obtained through* (3.10) *from an incoherent channel extension.*

Proof. Following [8], for any measurement $\{A_{i|\lambda}\}$ we can write:

$$\begin{aligned}\Lambda_{i|\lambda}^{C\to B}(\cdot) &= \mathrm{tr}_A\left[A_{i|\lambda}\Lambda^{C\to AB}(\cdot)\right] = \mathrm{tr}_A\left[A_{i|\lambda}\sum_k \Lambda_k^{C\to B}(\cdot)\otimes\rho_k\right] \\ &= \sum_k \Lambda_k^{C\to B}(\cdot)\,\mathrm{tr}_A\left[A_{i|\lambda}\rho_k\right] = \sum_k p_k(i|\lambda)\ \Lambda_k^{C\to B}(\cdot)\end{aligned} \tag{3.11}$$

Which proves that $\Lambda^{C\to AB}$ is unsteerable.
On the other hand, any set $\{\Lambda_{i|\lambda}^{C\to B} = \sum_k p_k(i|\lambda)\ \Lambda_k^{C\to B}\}$ of compatible instruments can be obtained applying the set of measurements $\{A_{i|\lambda} = \sum_k p_k(i|\lambda)\,|k\rangle\langle k|\}$ to the A output of $\Lambda^{C\to AB} = \sum_k \Lambda_k^{C\to B}\otimes|k\rangle\langle k|$. By

direct calculation we have:

$$\begin{aligned}\mathrm{tr}_A\left[A_{i|\lambda}\Lambda^{C\to AB}(\cdot)\right] =& \mathrm{tr}_A\left[\left(\sum_k p_k(i|\lambda)\left|k\right\rangle\left\langle k\right|\right)\cdot\left(\sum_{k'}\Lambda^{C\to B}_{k'}(\cdot)\otimes\left|k'\right\rangle\left\langle k'\right|\right)\right]\\ =&\sum_{k,k'}\delta_{k,k'}p_k(i|\lambda)\Lambda^{C\to B}_{k'}(\cdot)=\sum_k p_k(i|\lambda)\Lambda^{C\to B}_k(\cdot)\\ =&\Lambda^{C\to B}_{i|\lambda}\end{aligned}$$

□

Thus, if Bob can be convinced that Alice can steer his channel, then Alice must have access to a coherent extension of that channel.

3.2.1 Channel assemblages

In analogy with state assemblages, we can define channel assemblages as:

Definition 3.10. A *channel assemblage* $\{\Lambda^{C\to B}_{i|\lambda}\}$ is a set of weakly compatible instruments. That is, a set of instrument summing up to the same channel. The elements that compose each instrument are also called *sub-channels*.

Requiring the channels to be weakly compatible ensures that the non-signaling condition (3.3) is met for every input state.

Definition 3.11. A channel assemblage $\{\Lambda^{C\to B}_{i|\lambda}\}$ is said to be unsteerable if its instrument elements are compatible[5]. That is, if there is a joint instrument $\{\mathcal{G}_k\}$ such that $\Lambda^{C\to B}_{i|\lambda}=\sum_k p_k(i|\lambda)\mathcal{G}_k$.
A channel assemblage is steerable if it is not unsteerable.

[5] See definition 2.5

3.3 Steering quantification

The task of quantum steering requires a quantum resource, namely entanglement, to be shared by the two parties. However, as we saw in example 3.2, entanglement alone is not sufficient for steering. We are therefore led to think about the steerability of a state as a resource in itself, only partially connected to its entanglement.
This approach prompts us to quantify the amount of steering that can be carried out using a particular shared state, so that one could define the 'steerability' of a state independently from how much that state is entangled.
However, deciding the steerability of a particular shared state is in general difficult, since one has to consider all possible measurements that Alice could use to attempt steering. A commonly used approach is to quantify the steerability of state assemblages. We will find that the task of steering quantification is easier for a state assemblage than for the corresponding shared state, as the former contains the information about which measurement strategy Alice is using.
Several steering quantifiers that use this approach have been proposed. In the following discussion we will present two relevant examples.

3.3.1 Steerable weight

The simplest approach to steering quantification is represented by the notion of *steerable weight*, as introduced by [9]. For any assemblage $\mathcal{A} = \{\rho_{a|\lambda}\}$, we consider the decomposition:

$$\rho_{a|\lambda} = \mu \ \rho^{S}_{a|\lambda} + (1-\mu) \ \rho^{US}_{a|\lambda} \qquad \forall \ a, \lambda \tag{3.12}$$

where the superscripts denote respectively an unsteerable assemblage and a steerable one.

Definition 3.12. Given a state assemblage $\{\rho_{a|\lambda}\}$, we identify its *steerable weight* $\tilde{\mu}$ as the minimum value of μ for which the decomposition (3.12) exists.

Intuitively, this quantifier measures the fraction of steerable assemblages required to construct $\mathcal{A}$.

Proposition 3.13. *The steerable weight of an assemblage* $\mathcal{A} = \{\rho_{a|\lambda}\}$ *can be efficiently calculated with an SDP [9]. Denoting* $\tilde{\mu} = 1 - \mu^*$*, we can obtain* μ^* *as:*

$$\begin{aligned} \max \quad & \mathrm{tr}\left[\sum_k \rho_k\right] \\ \text{s.t.} \quad & \rho_{a|\lambda} - \sum_k p_k(a|\lambda)\rho_k \geq 0 \quad \forall a, \lambda \\ & \rho_k \geq 0 \quad \forall k \end{aligned} \tag{3.13}$$

Proof. From the definitions of steerable weight and steerability of state assemblages we get the following constraints for the maximization of μ^* (which is equivalent to the minimization of μ):

i) $\rho_{a|\lambda} = (1-\mu^*)\ \rho^S_{a|\lambda} + \mu^*\ \rho^{US}_{a|\lambda} \qquad \forall\, a, \lambda$

ii) $\rho^{US}_{a|\lambda} = \sum_k p_k(a|\lambda)\rho_k \qquad \forall\, a, \lambda$

iii) $\mathrm{tr}\left[\sum_k \rho_k\right] = 1, \quad \rho_k \geq 0 \qquad \forall k$

iv) $\sum_a \rho^S_{a|\lambda} = \rho \quad \forall \lambda$

v) $\rho^S_{a|\lambda} \geq 0 \qquad \forall a, \lambda$

Using $i)$ and $ii)$ the last constraint can be rewritten as:

$$\rho^S_{a|\lambda} = \frac{1}{1-\mu^*}\left(\rho_{a|\lambda} - \mu^* \sum_k p_k(a|\lambda)\rho_k\right) \geq 0 \tag{3.14}$$

This also automatically guarantees that $iv)$ is satisfied, as long as the input assemblage is valid. Defining the new variables $\tilde{\rho_k} = \mu^* \rho_k$, condition $iii)$ is equivalent to $\sum_k \tilde{\rho_k} = \mu^*, \quad \tilde{\rho_k} \geq 0 \, \forall k$. This, together with the assumption $0 < \mu^* < 1$ allows us to write (3.14) as:

$$\rho_{a|\lambda} - \sum_k p_k(a|\lambda)\tilde{\rho_k} \geq 0 \tag{3.15}$$

With the identification $\tilde{\rho_k} \to \rho_k$, and remembering that $\sum_k \tilde{\rho_k} = \mu^*$ we obtain the required SDP (3.13). □

Remark 3.14. This quantifier suffers from the problem that any assemblage arising from a shared pure state is always maximally steerable [9], regardless of the amount of entanglement present in the state.

3.3.2 Steering robustness

The notion of *steering robustness* was introduced in [10] and can be defined both for states and for assemblages. The idea behind it is to measure how much noise one needs to add in order to spoil steerability. As in the case of the steerable weight, it can be shown that the steering robustness can be calcultated as an SDP [10].

Definition 3.15. Given an assemblage $\mathcal{A} = \{\rho_{a|\lambda}\}$, its *steering robustness* $R(\mathcal{A})$ is:

$$R(\mathcal{A}) = \min\left\{t \mid t \geq 0,\ \frac{\rho_{a|\lambda} + t\ \tau_{a|\lambda}}{t+1} \text{ is unsteerable,} \tau_{a|\lambda} \text{ is an assemblage}\right\} \tag{3.16}$$

Since state assemblages on Bob's side can be prepared by Alice by the use of a certain measurement set $\{A_{i|\lambda}\}$ on the joint state ρ_{AB}, the definition of this quantifier has been extended to the shared state.

Definition 3.16. For a given shared state ρ_{AB}, we define its *steering robustness* as:

$$R^{A\to B}(\rho_{AB}) = \sup_{\mathcal{A}\in\Omega} R(\mathcal{A}) \tag{3.17}$$

Where Ω is the set of all Bob's state assemblages corresponding to any possible measurement strategy Alice may use.

In the following chapter, we will use these quantifiers to quantify the incompatibility of a set of generalized measurements.

Chapter 4

Joint Measurability and Steering

In this chapter we will focus on describing the connections that have been found between the steerability of a scenario and the incompatibility of the measurement used to complete the task. We will show that every steerability problem can be thought as a joint measurability problem and vice versa.

4.1 The general connection

In the preceeding chapters the notions of joint mesurability (or compatibility) and steering have been introduced. The two concept are however deeply related. Consider a typical steering scenario where two parties, Alice and Bob, share a bipartite state ρ_{AB}. Alice has access to

a set of POVMs on her system $\{A_{a|\lambda}\}$; she chooses to perform one of the available measurements (labeled by λ) and obtains an outcome a. This leaves the state of Bob's system in the unnormalized partial state $\rho_{a|\lambda} = \mathrm{tr}_A[(A_{a|\lambda} \otimes \mathbb{1})\ \rho_{AB}]$, with the non-signaling condition requiring that $\sum_a \rho_{a|\lambda} = \rho_B \ \ \forall\lambda$, where ρ_B is the total reduces state for Bob.
Following [11], we define $\Pi_B : \mathcal{H}_B \to \mathcal{K}_B \subset \mathcal{H}_B$ as the projection on the subspace of $\mathcal{H}_B$ spanned by ρ_B, *i.e.* $\mathcal{K}_B = \mathrm{range}(\rho_B)$ and $\Pi_B\ \Pi_B^\dagger = \mathbb{1}_{\mathcal{K}_B}$. Since $\rho_{a|\lambda}$ are positive operators, the non-signaling condition implies $\mathrm{range}(\rho_{a|\lambda}) \subset \mathrm{range}(\rho_B)$. We then define the restrictions $\tilde{\rho}_{a|\lambda} = \Pi_B \rho_{a|\lambda} \Pi_B^\dagger$ and $\tilde{\rho}_B = \Pi_B \rho_B \Pi_B^\dagger$, preserving the positivity of the operators.
Using the restricted states we define the so called of Bob's *steering-equivalent* (SE) observables $B_{a|\lambda} \in \mathcal{L}(\mathcal{K}_{\rho_B})$ as:

$$B_{a|\lambda} = (\tilde{\rho}_B)^{-\frac{1}{2}}\ \tilde{\rho}_{a|\lambda}\ (\tilde{\rho}_B)^{-\frac{1}{2}} \tag{4.1}$$

These operators are positive, and the non-signaling condition ensures that $\sum_a B_{a|\lambda} = \mathbb{1}_{\mathcal{K}_B}\ \forall\lambda$. Therefore, for each λ, they form a POVM.

Theorem 4.1. *The state assemblage*[1] $\{\rho_{a|\lambda}\}$ *is unsteerable if and only if the set of POVMs* $\{B_{a|\lambda}\}$ *is jointly measurable.*

Proof. The unsteerability of the assemblage $\{\rho_{a|\lambda}\}$ is equivalent to the existence of a LHS for $\rho_{a|\lambda}$, that is:

$$\rho_{a|\lambda} = \sum_k p_k(a|\lambda)\rho_k \tag{4.2}$$

If such a model exist, a joint observable for $\{B_{a|\lambda}\}$ can easily be constructed through relation (4.1). The same relation can also be used to

[1] A definition of state assemblage is given in section 3.3

prove the other direction, going from a joint observable for $\{B_{a|\lambda}\}$ to a LHS model for $\{\rho_{a|\lambda}\}$. □

Therefore, every steerability problem can be cast into a joint measurability problem and vice versa (taking $\rho_B = \frac{\mathbb{1}}{d}$). An explicit example of Bob's SE observables is given in example 4.3.

4.1.1 Interpretation of SE observables

A simple interpretation of Bob's steering-equivalent observables can be found if the shared state is pure. Let $\rho = \sum_{i,j}^{n} \lambda_i \lambda_j \left|ii\right\rangle \left\langle jj\right|$, where $\{\left|i\right\rangle_A\}_1^{d_A}$ and $\{\left|i\right\rangle_B\}_1^{d_B}$ are the local bases associated with the Schmidt decomposition of ρ ($n \leq \min(d_A, d_b)$, $\lambda_i \geq 0 \quad \forall i$ and $\mathrm{tr}[\rho] = \sum_i \lambda_i^2 = 1$). In this basis the reduced states for Alice and Bob can both be written as $\rho_X = \sum_i \lambda_i^2 \left|i\right\rangle \left\langle i\right|_X$, with $X = A, B$. Their ranges $\mathcal{K}_A$ and $\mathcal{K}_B$ are therefore isomorphic with the identification $\left|i\right\rangle_A \leftrightarrow \left|i\right\rangle_B$. We can now write:

$$\begin{aligned}
\rho_{a|\lambda} &= \mathrm{tr}[(A_{a|\lambda} \otimes \mathbb{1})\rho] = \sum_{ij} \lambda_i \lambda_j \, \mathrm{tr}_A[(A_{a|\lambda} \otimes \mathbb{1}) \left|ii\right\rangle \left\langle jj\right|] \\
&= \sum_{ij} \lambda_i \lambda_j \, \mathrm{tr}_A \left[A_{a|\lambda} \left|i\right\rangle \left\langle j\right|\right] \left|i\right\rangle \left\langle j\right| = \sum_{ij} \lambda_i \lambda_j \left\langle j\right| A_{a|\lambda} \left|i\right\rangle \left|i\right\rangle \left\langle j\right| \\
&= \sum_{ij} \lambda_i \lambda_j \left|i\right\rangle \left\langle i\right| A_{a|\lambda}^T \left|j\right\rangle \left\langle j\right| = \rho_A^{1/2} A_{a|\lambda}^T \rho_A^{1/2}
\end{aligned} \tag{4.3}$$

We find the following form for Bob's SE observables:

$$B_{a|\lambda} = \tilde{\rho}_B^{-1/2} \; \rho_A^{1/2} \; A_{a|\lambda}^T \; \rho_A^{1/2} \; \tilde{\rho}_B^{1/2} \tag{4.4}$$

With the above identification ($|i\rangle_A \leftrightarrow |i\rangle_B$) in mind, we have that $\rho_A \simeq \rho_B$, and therefore:

$$B_{a|\lambda} \simeq A^T_{a|\lambda} \tag{4.5}$$

This equivalence relation holds only in the restrictions $\mathcal{K}_A$ and $\mathcal{K}_B$ of $\mathcal{H}_A$ and $\mathcal{H}_B$. In these restrictions, Bob's SE observables are just (up to a change of basis) Alice POVMs, and therefore incompatibility of Alice (restricted) observables is enough to show steering if the shared state is pure.

Example 4.1. Consider the maximally entangled state $\rho_{AB} = |\psi_+\rangle\langle\psi_+| = \frac{1}{d} = \sum_{i,j=1}^{d} |ii\rangle\langle jj|$. Following [1], we obtain Bob's assemblage as:

$$\rho_{a|\lambda} = \mathrm{tr}_A \left[(A_{a|\lambda} \otimes \mathbb{1}) |\psi_+\rangle\langle\psi_+| \right] = \frac{1}{d} A^T_{a|\lambda} \tag{4.6}$$

It is now easy to see that the steerability of the assemblage $\{\rho_{a|\lambda}\}$ is equivalent to the joint measurability of Alice's measurements $\{A_{a|\lambda}\}$, since a LHV model for $\{\rho_{a|\lambda}\}$ automatically gives a joint observable for $\{A_{a|\lambda}\}$ through (4.6).

4.2 Incompatibilty quantifiers

With the above connection between incompatibility and steering in mind, we can extend the concept of steering quantification to incompatibility. The notions of *steerable weight* and *steering robustness* can then be used to define quantifiers for incompatibility.

4.2.1 Incompatibility weight

This quantifier has been introduced in [12], and it extends the steering quantifier presented in subsection 3.3.1.

Definition 4.2. Given a set of POVMs $\{M_{a|\lambda}\}$, their *incompatibility weight* is the minimum μ such that the following decomposition is admissible:

$$M_{a|\lambda} = \mu\ O_{a|\lambda} + (1-\mu)\ N_{a|\lambda} \tag{4.7}$$

$\{N_{a|\lambda}\}$ and $\{O_{a|\lambda}\}$ are set of POVMs, and the set $\{O_{a|\lambda}\}$ is jointly measurable.

Remark 4.3. This quantifier suffers from a problem relating to the one outlined in remark 3.14 for steerable weight. That is, when the elements of the set are rank-1 operators (*i.e.* the measurements are PVMs) this quantifier is always maximal. Therefore, pairs of arbitrary close PVMs are always maximally incompatible according to IW, in the same way as state assemblages arising from pure bipartite states are maximally steerable according to SW (even for for arbitrary small entanglement). In the next section we present a quantifier that solves this problem.

4.2.2 Incompatibility robustness

Following the idea of *steering robustness*, introduced in subsection 3.3.2, we now present the notion of *incompatibility robustness* [11].

Definition 4.4. Given a set of POVMs $\{M_{a|\lambda}\}$, their *incompatibility robustness* is the minimum t such that the set:

$$\{O_{a|\lambda}\} = \left\{ \frac{M_{a|\lambda} + t\ N_{a|\lambda}}{t+1} \right\} \tag{4.8}$$

is jointly measurable ($\{N_{a|\lambda}\}$ is a general set of POVMs).

It can be thought as a quantification of the minimal (due to the optimization over $\{N_{a|\lambda}\}$) amount of noise one has to add in order to spoil incompatibility.

As for the case of steering quantifiers, the aforementioned incompability quantifiers can be efficiently calculated by the use of SDPs.

4.3 Steering inequalities from incompatibility

In light of theorem 4.1, previously found incompatibility inequalities can be used to devise steering inequalities for the characterization of steerable scenarios [11].

Example 4.2. Consider the qubit assemblage $\{\rho_{a|\lambda}\}$, with $a = \pm$ and $\lambda \in \{1, 2\}$:

$$\rho_{\pm|\lambda} = t^{\pm}_{\lambda}\mathbb{1} + \vec{s}^{\pm}_{\lambda} \cdot \vec{\sigma}; \quad \vec{s}^{\pm}_{\lambda} = (x^{\pm}_{\lambda}, y^{\pm}_{\lambda}, z^{\pm}_{\lambda}) \tag{4.9}$$

In order for steering to be possible, the assemblage must correspond to a state $\rho_B = \sum_a \rho_{a|\lambda}\ \forall\lambda$ with rank 2, otherwise the total shared state would be separable (pure marginal states can only be obtained if the total state is separable).
For this assemblage, the corresponding SE observables are given by:

$$B_{+|\lambda} = \frac{1}{2}((1 + \alpha_\lambda)\mathbb{1} + \vec{r}_\lambda \cdot \vec{\sigma}); \quad B_{-|\lambda} = \mathbb{1} - B_{+|\lambda} \tag{4.10}$$

Where α_λ and $\vec{r}_\lambda$ are chosen so that inverting equation (4.1) gives the desired assemblage $\{\rho_{a|\lambda}\}$ (the details of the derivation can be found in

[11]).
In chapter 2, we shoved that such observables are jointly measurable if they satisfy relation 2.16; the incompatibility of observables (4.10) can then be quantified[2] by the amount of violation of the following inequality:

$$\|\vec{r_1} + \vec{r_2}\| + \|\vec{r_1} - \vec{r_2}\| \leq 2 \tag{4.11}$$

This is a necessary condition for joint measurability, thus, if the SE observables are found to violate such inequality, we can conclude that they are incompatible and therefore (using theorem 4.1) that the assemblage $\{\rho_{a|\lambda}\}$ is steerable.

As shown in example 4.2, we can use known incompatibility relations to construct inequalities and quantifiers for steering. In fact, [11] gives a refined version of condition 4.11 which is both necessary and sufficient in order to have incompatibility of the SE observables. Therefore, the amount of violation of such inequality can be used as a direct quantification of the steerability of the assemblage. In the same paper it is also given another example of such inequalities that holds for the case of three dichotomic observables on a qubit system.

Example 4.3. We wish to use the criterion presented in example 4.2 to show the steerability of a state assemblage. For the sake of simplicity, let us assume that Alice and Bob share the maximally entangled state $|\psi_+\rangle = \frac{1}{\sqrt{2}} |00\rangle + |11\rangle$ and that Alice steering attemps are made using projective measurements along the z and x axis (that is, using the PVMs $\frac{1}{2}(\mathbb{1} \pm \sigma_z)$ and $\frac{1}{2}(\mathbb{1} \pm \sigma_x)$).

[2]This quantifier is known as *incompatibility degree* and has been introduced by [13]

It is easy to see that the state assemblage for Bob will be:

$$\begin{aligned} \rho_{+|z} &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad & \rho_{+|x} &= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \rho_{-|z} &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad & \rho_{-|x} &= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned} \tag{4.12}$$

Using the notation presented in section 4.1, we now write:

$$\rho_B = \rho_{+|z} + \rho_{-|z} = \rho_{+|x} + \rho_{-|x} = \frac{1}{2}\mathbb{1} \tag{4.13}$$

Since ρ_B is clearly rank 2, we have that $\tilde{\rho_B} = \rho_B$ and $\tilde{\rho_{a|\lambda}} = \rho_{a|\lambda}$; we also have $\rho_B^{-\frac{1}{2}} = \sqrt{2}\,\mathbb{1}$. Using equation (4.1) we find Bob's SE observables to be:

$$\begin{aligned} B_{+|z} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad & B_{+|x} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ B_{-|z} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad & B_{-|x} &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned} \tag{4.14}$$

The parameters α_λ, $\vec{r}_\lambda$ of example 4.2 are then found to be:

$$\begin{aligned} \alpha_z &= 0; \quad & \vec{r}_z &= (0,0,1) \\ \alpha_x &= 0; \quad & \vec{r}_x &= (1,0,0) \end{aligned} \tag{4.15}$$

In this case, equation (4.11) becomes:

$$\|\vec{r_x} + \vec{r_z}\| + \|\vec{r_x} - \vec{r_z}\| = \sqrt{2} + \sqrt{2} = 2\sqrt{2} > 2 \tag{4.16}$$

We conclude that Bob's SE observable are incompatible, and therefore, by theorem 4.1, that the assemblage is steerable.

Remark 4.5. Since the state shared between the parties is pure, as expected from the results of subsection 4.1.1, we find that Bob's SE observables found in equation (4.14) match exactly Alice's PVMs.

4.4 Channel steering and joint measurability

4.4.1 Incompatibility of Alice's measurements

In [14] a result linking the steerability of a channel extension to the incompatibility of Alice's measurements is presented. However, it is possible to find a counterexample to this claim. The proposed link is:

Claim 4.6. *The channel assemblage* $\{\Lambda_{a|\lambda}^{C\to B}\}$ *for any channel* $\Lambda^{C\to B}$ *with* $\Lambda_{a|\lambda}^{C\to B} = \mathrm{tr}_A[A_{a|\lambda}\Lambda^{C\to AB}]$ *is unsteerable for any channel extension* $\Lambda^{C\to AB}$ *of* $\Lambda^{C\to B}$ *if and only if the set of POVMs* $\{A_{a|\lambda}\}$ *applied by Alice is jointly measurable.*

Proposition 4.7. *To provide a counterexample to claim 4.6, we will prove that there exist a channel that admits only incoherent extensions, and therefore that any channel assemblage for Bob output must be unsteerable regardless of Alice choice of measurement.*

Proof. This proof holds for any channel $\Lambda^{C\to B}$ that maps pure states into pure states. For simplicity we will consider $\mathcal{H}_C = \mathcal{H}_B$ and take $\Lambda^{C\to B}$ as the identity channel, so that $\Lambda^{C\to B}(\rho) = \rho$. All extensions of this channel must then satisfy:

$$\mathrm{tr}_A[\Lambda^{C\to AB}(\rho)] = \rho \tag{4.17}$$

If ρ is a pure state, say $\rho = |\psi\rangle\langle\psi|$, equation (4.17) implies that the bipartite state $\Lambda^{C\to AB}(\rho)$ is separable, since it is well known that if the marginal of a multipartite state is pure, then the multipartite state must be separable.
If ρ is mixed, say $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, then we have:

$$\Lambda^{C\to AB}(\rho) = \Lambda^{C\to AB}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) = \sum_i p_i \Lambda^{C\to AB}\left(|\psi_i\rangle\langle\psi_i|\right) \tag{4.18}$$

And since $\Lambda^{C\to AB}(|\psi_i\rangle\langle\psi_i|)$ must be separable, we can write:

$$\Lambda^{C\to AB}(\rho) = \sum_i p_i \sum_j \mu_{ij}\rho_{ij}^A \otimes \rho_{ij}^B = \sum_k \lambda_k \rho_k^A \otimes \rho_k^B \tag{4.19}$$

where we defined the multiindex $k = (i,j)$ and $\lambda_{k=(i,j)} = p_i\mu_{ij}$.
Therefore we proved that the state $\Lambda^{C\to AB}(\rho)$ is separable for any input matrix ρ and for any channel extension $\Lambda^{C\to AB}$.
Hence, all extensions $\Lambda^{C\to AB}$ of the identity channel are incoherent and therefore unsteerable regardless of Alice choice of measurement. □

A modified version of claim 4.6 can however be proved:

Theorem 4.8. *The channel assemblage $\{\Lambda_{a|\lambda}\}$ arising from the minimal dilation*[3] *of the channel Λ as $\Lambda_{a|\lambda} = \mathrm{tr}_A[(A_{a|\lambda}\otimes \mathbb{1})\ V(\rho\otimes\mathbb{1})V^\dagger]$ is unsteerable if and only if the set of POVMs $\{A_{a|\lambda}\}$ is jointly measurable.*

Proof. We will first prove that unsteerability implies joint measurability. We have:

$$\Lambda_{a|\lambda} = \sum_k p_k(a|\lambda)\Lambda_k \tag{4.20}$$

[3]See proposition 2.9 and corollary 2.10 for details

Summing over a on both sides, and using the weak compatibility of $\{\Lambda_{a|\lambda}\}$ we find:

$$\Lambda = \sum_k \sum_a p_k(a|\lambda)\Lambda_k = \sum_k \Lambda_k \tag{4.21}$$

that is, the instrument $\{\Lambda_k\}$ is weakly compatible with $\{\Lambda_{a|\lambda}\}$. Using proposition 2.9, we can find a unique POVM $\{G_k\}$ such that:

$$\Lambda_k = \mathrm{tr}_A[(G_k \otimes \mathbb{1})V(\rho \otimes \mathbb{1})V^\dagger] \tag{4.22}$$

Putting (4.20) and (4.22) together we have:

$$\begin{aligned}\mathrm{tr}_A[(A_{a|\lambda} \otimes \mathbb{1})\ V(\rho \otimes \mathbb{1})V^\dagger] = \Lambda_{a|\lambda} &= \sum_k p_k(a|\lambda)\Lambda_k \\ &= \sum_k p_k(a|\lambda)\,\mathrm{tr}_A[(G_k \otimes \mathbb{1})V(\rho \otimes \mathbb{1})V^\dagger] \\ &= \mathrm{tr}_A[(\sum_k p_k(a|\lambda)G_k \otimes \mathbb{1})V(\rho \otimes \mathbb{1})V^\dagger]\end{aligned} \tag{4.23}$$

Using proposition 2.9 again, we have that the POVMs $\{A_{a|\lambda}\}$ must be unique, hence:

$$A_{a|\lambda} = \sum_k p_k(a|\lambda)G_k \tag{4.24}$$

that is, $\{G_k\}$ is a joint observable for $\{A_{a|\lambda}\}$.
The inverse implication (joint measurability $\Rightarrow$ unsteerability) is trivial, as the existence of a joint observable for $\{A_{a|\lambda}\}$ directly implies the existence of a set of instruments $\{\Lambda_k\}$ such that $\{\Lambda_{a|\lambda} = \sum_k p_k(a|\lambda)\Lambda_k\}$ through the defining relation of $\Lambda_{a|\lambda}$.

□

Example 4.4. Let us apply this result to the identity channel used in proposition 4.7. We find that the minimal dilation of the identity channel $\mathbb{1}^{C\to B} : \mathcal{H}_C \to \mathcal{H}_B$ is constructed on the extended space $\mathcal{H}_B \otimes \mathbb{C}_A$, that is Alice's output space is just complex numbers. This is due to the fact that the dimension of the minimal dilation is equal to the number of linearly independent operators in the Kraus decomposition of the channel[15]. In this case, all possible measurement on Alice's subsystem would be jointly measurable (since matrix multiplication reduces to the product of complex numbers), and therefore no steering would be possible.

Example 4.5. We will now give an example of a steerable channel assemblage constructed from a qubit quantum channel.
The defining relation of the channel $\Lambda^{C\to B} : \mathbb{C}^2 \to \mathbb{C}^2$ is given as:

$$\Lambda^{C\to B} \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{4.25}$$

We can check that this relation defines a valid channel by finding a Kraus decomposition for it and using theorem 1.10. We find that the channel (4.25) is indeed given by:

$$\Lambda^{C\to B}(\rho) = |0\rangle\langle 0|\,\rho\,|0\rangle\langle 0| + |1\rangle\langle 1|\,\rho\,|1\rangle\langle 1| \tag{4.26}$$

That is, under the action of the channel, the states $|0\rangle$ and $|1\rangle$ are left untouched, while any other state is projected on the computational basis. A possible extension of this channel is given by the broadcast channel

$\Lambda^{C\to AB} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, defined by:

$$\Lambda^{C\to AB}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} := \begin{pmatrix} a & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^* & 0 & 0 & b \end{pmatrix} \tag{4.27}$$

We can easily check that $\mathrm{tr}_A[\Lambda^{C\to AB}] = \Lambda^{C\to B}$. This broadcast channel can be implemented using a CNOT gate, using the quantum circuit depicted in figure 4.1, with $|\psi\rangle$ as input state.

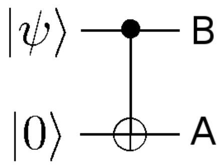


FIGURE 4.1: The 'CNOT' broadcast channel

Imagine that Alice uses the set of measurements:

$$\{S_{z|\pm} = \frac{1}{2}(\mathbb{1} \pm \sigma_z), \quad S_{x|\pm} = \frac{1}{2}(\mathbb{1} \pm \sigma_x)\} \tag{4.28}$$

Using proposition 1.21, we can construct the final bipartite state after Alice's measurement. Following the relation $\Lambda^{C\to B}_{a|\pm} = \mathrm{tr}_A[(S_{a|\pm} \otimes \mathbb{1})\Lambda^{C\to AB}]$, we trace away Alice's subsystem and find the channel assemblage for Bob:

$$\begin{aligned} &\Lambda^{C\to B}_{z|+}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}; \quad \Lambda^{C\to B}_{x|+}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a & c \\ c^* & d \end{pmatrix} \\ &\Lambda^{C\to B}_{z|-}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}; \quad \Lambda^{C\to B}_{x|-}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a & c \\ c^* & d \end{pmatrix} \end{aligned} \tag{4.29}$$

We conclude that this is a steerable assemblage, for example, the input state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ gives the maximally entangled $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ state as output, allowing Alice to steer Bob. With this input, the state assemblage on Bob's output is the same discussed in example 4.3.

Remark 4.9. Note that summing over the outcomes $+, -$ for each of Alice's measurement choices we obtain the same total channel, as required by the non signaling condition.

Remark 4.10. This channel assemblage gives rise to steerable state assemblages only for some inputs states. For example, if the input state is $|0\rangle$, the bipartite output state is $|00\rangle$ which, being separable, does not allow steering.

4.4.2 Bob's SE observables for channel steering

The notion of Bob's steering equivalent observables, introduced for state steering in section 4.1, can be easily extended to a channel steering scenario:

Proposition 4.11. *The channel extension $\Lambda^{C\to AB}$ is unsteerable if and only if the set of POVMs $\{B_{a|\lambda} := (\tilde{\rho}_B)^{-\frac{1}{2}}\ \rho_{a|\lambda}\ (\tilde{\rho}_B)^{-\frac{1}{2}}\}$, where $\rho_B = \sum_a \rho_{a|\lambda}\ \ \forall\lambda$ and*[4] $\rho_{a|\lambda} := J(\Lambda^{C\to B}_{a|\lambda}) = J(\mathrm{tr}_A[(A_{a|\lambda}\otimes \mathbb{1})\Lambda^{C\to AB}])$, *is jointly measurable.*

Proof. Using the results in [8] it can be shown that the steerability of the channel extension $\Lambda^{C\to AB}$ is equivalent to the steerability of the state

[4] $J(\Lambda^{C\to B}_{a|\lambda})$ denotes the state assemblage obtained through the Choi-Jamiolkoski isomorphism from the channel assemblage $\Lambda^{C\to B}_{a|\lambda}$

assemblage $\{\rho_{a|\lambda} := J(\Lambda_{a|\lambda}^{C\to B})\}$.
Defining then the steering-equivalent observables:

$$B_{a|\lambda} := (\rho_B)^{-\frac{1}{2}}\ \rho_{a|\lambda}\ (\rho_B)^{-\frac{1}{2}}$$

as we did in (4.1) and using Theorem 4.1, we have that the joint measurability of $\{B_{a|\lambda}\}$ is equivalent to the steerability of $\{\rho_{a|\lambda}\}$ and therefore to the steerability of the channel extension. □

Example 4.6. As an example of proposition 4.11, we will explicitly show that the Bob's observables arising from the identity channel through proposition 4.11 are always jointly measurable.
The channel assemblage for Bob is:

$$\Lambda_{a|\lambda}^{C\to B} = \mathrm{tr}_A[(A_{a|\lambda} \otimes \mathbb{1})\Lambda^{C\to AB}] \tag{4.30}$$

Since we are taking $\Lambda^{C\to B}$ to be the identity channel, using the arguments of proposition 4.7 we find that any possible extension $\Lambda^{C\to AB}$ must be incoherent. With this in mind, using the definition of incoherent channel extension, we can write the state assemblage $\rho_{a|\lambda} = J(\Lambda_{a|\lambda}^{C\to B})$ as:

$$\begin{aligned}
\rho_{a|\lambda} =& \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \mathrm{tr}_A[(A_{a|\lambda}\otimes \mathbb{1})\left(\sum_k \sigma_k \otimes \Lambda_k^{C\to B}(|i\rangle\langle j|)\right)] \\
=& \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \sum_k \mathrm{tr}_A[(A_{a|\lambda}\otimes \mathbb{1})\ \left(\sigma_k \otimes \Lambda_k^{C\to B}(|i\rangle\langle j|)\right)] \\
=& \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \sum_k \mathrm{tr}_A[(A_{a|\lambda}\sigma_k]\ \Lambda_k^{C\to B}(|i\rangle\langle j|) \\
=& \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \sum_k p_k(a|\lambda)\ \Lambda_k^{C\to B}(|i\rangle\langle j|) \\
=& \frac{1}{d}\sum_k p_k(a|\lambda) \sum_{i,j} |i\rangle\langle j| \otimes\ \Lambda_k^{C\to B}(|i\rangle\langle j|) \\
=& \frac{1}{d}\sum_k p_k(a|\lambda)\rho_k
\end{aligned} \tag{4.31}$$

We have that the state assemblage $\rho_{a|\lambda}$ can be written as a LHS model for Bob, and is therefore unsteerable. Hence, using theorem 4.1 we have that $G_k = \tilde{\rho}_B^{-\frac{1}{2}}\rho_k\tilde{\rho}_B^{-\frac{1}{2}}$ is a joint observable for $B_{a|\lambda}$

Chapter 5

Resource theory of incompatibility

As we've seen in the case of steering, measurement incompatibility is needed in order to show the non classical properties of a quantum system. Since the set of measurements available to extract information from a system is often limited by the experimental setup, the description of incompatibility as a quantum resource is a natural step to take.
Resource theories are generally characterized by two main components: the physical property to be treated as a resource and a class of operations where the property plays the role of a useful resource. These operations are called the free operations, and they generally represent the class of actions easily implementable in a real scenario. Free operations are defined by the requirement of mapping every free state (*i.e.* every state without the property) into a free state, that is, they don't create the resource. Resource theories of this type have been developed for many quantum

phenomena, like entanglement [16, 17], nonlocality [18, 19], and steering [20].
In our case, we will identify the free operations as the action of Heisenberg-picture quantum channels on POVMs. The idea behind this choice is that in any physical setup, the measured system will undergo some kind of transformation before interacting with the measuring device, be it caused by noise or by the design of the experiment. We can therefore describe this state transformation as a quantum channel applied to the measurement, acting on the non-transformed state.

5.1 Theoretical framework

In the introduction of this chapter, the idea of choosing quantum channels as the free operations for the theory was motivated by physical considerations about the nature of measurements devices in real experiments. From the theoretical point of view, this choice is supported by the following result:

Proposition 5.1. *Let* $\Lambda : T(\mathcal{H}) \to T(\tilde{\mathcal{H}})$ *be a quantum channel in the Heisenberg picture and* $\{E_{i|\lambda}\}$ *a set of POVMs. Then, if* $\{E_{i|\lambda}\}$ *is jointly measurable, the set* $\{\Lambda(E_{i|\lambda})\}$ *is also jointly measurable*

Proof. From the definition of joint measurability we know that it exists a joint observable $\{G_k\}$ and positive probabilities $p_k(i|\lambda)$ such that $E_{i|\lambda} =$

$\sum_k p_k(i|\lambda)G_k$. Thus:

$$\begin{aligned}\Lambda(E_{i|\lambda}) &= \Lambda\left(\sum_k p_k(i|\lambda)G_k\right) = \sum_k \Lambda\left(p_k(i|\lambda)G_k\right) \\ &= \sum_k p_k(i|\lambda)\Lambda(G_k)\end{aligned} \tag{5.1}$$

and therefore we find that $\{\Lambda(G_k)\}$ is a joint observable for $\{\Lambda(E_{i|\lambda})\}$. □

Remark 5.2. For the rest of this chapter, all channels will be taken to be in the Heisenberg picture, unless otherwise stated.

Proposition 5.1 shows that quantum channels map any free state (*i.e.* any set of jointly measurable observables) into a free state, therefore, we can say that quantum channels don't create the resource, *i.e* incompatibility.

Definition 5.3. A function $\mathcal{I}$ mapping sets of observables into non-negative reals $\mathbb{R}_{\geq 0}$ is an *incompatibility monotone* [21] if it satisfies the following requirements:

- $\mathcal{I}\left(\{E_{i|\lambda}\}\right) = 0$ for all jointly measurable $\{E_{i|\lambda}\}$
- $\mathcal{I}$ does not increase under the action of a quantum channel, *i.e*

$$\mathcal{I}\left(\{E_{i|\lambda}\}\right) \geq \mathcal{I}\left(\{\Lambda(E_{i|\lambda})\}\right) \tag{5.2}$$

  where Λ is any quantum channel.

The first requirements embodies the fact that we expect all jointly measurable sets to have zero incompatibility, while the second formalizes the idea that incompatibility should not be created under the action of a free operation.

Definition 5.4. An incompatibility monotone $\mathcal{I}$ is a *convex incompatibility monotone* if the following holds: given a real number $0 \leq \alpha \leq 1$ and the two sets $\{E_{i|\lambda}\}$ and $\{F_{i|\lambda}\}$, we have:

$$\mathcal{I}\left(\{\ \alpha\ E_{i|\lambda} + (1-\alpha)\ F_{i|\lambda}\ \}\right) \leq \alpha\ \mathcal{I}\left(\{E_{i|\lambda}\}\right) + (1-\alpha)\ \mathcal{I}\left(\{F_{i|\lambda}\}\right) \quad (5.3)$$

This last requirement states that incompatibility should not increase when probabilistically mixing two sets of observables.

Example 5.1. Consider an incompatibility monotone $\mathcal{I}$ and a set of incompatible observables, say $\{E_{\pm|\lambda} = \frac{1}{2}(\mathbb{1} \pm \sigma_\lambda)\}_{\lambda \in x,z}$. Take now a set of compatible observables which can be thought of as noise, say $\{F_{\pm|\lambda} = \frac{1}{2}\mathbb{1}\}_{i|\lambda}$.
It is possible to choose a strictly positive value for α so that the measurements $\{\ \alpha\ E_{i|\lambda} + (1-\alpha)\ F_{i|\lambda}\ \}$ are compatible[1], making the left hand side of (5.3) to be equal to 0. The right hand side, however, can be greater than zero, since $\{E_{i|\lambda}\}$ are incompatible.

Proposition 5.5. *Let $\{E_{i|\lambda}\}$ and $\{F_{i|\lambda}\}$ be two set of measurement connected by a unitary transformation, so that $E_{i|\lambda} = U^\dagger F_{i|\lambda} U \quad \forall i, \lambda$.*
Then, given any incompatibility monotone $\mathcal{I}$, it must hold that $\mathcal{I}(\{E_{i|\lambda}\}) = \mathcal{I}(\{F_{i|\lambda}\})$.

Proof. Define the unitary channel $\Lambda(\cdot) = U^\dagger \cdot U$, then we have that $E_{i|\lambda} = \Lambda(F_{i|\lambda}) \quad \forall i, \lambda$. Therefore, using the second property of incompatibility monotones:

$$\mathcal{I}(\{E_{i|\lambda}\}) \geq \mathcal{I}(\{\Lambda(E_{i|\lambda})\}) = \mathcal{I}(\{F_{i|\lambda}\}) \quad (5.4)$$

[1]For the explicit calculation, we refer to example 2.2

but we also have:

$$\mathcal{I}(\{F_{i|\lambda}\}) \geq \mathcal{I}(\{\Lambda^{-1}(F_{i|\lambda})\}) = \mathcal{I}(\{E_{i|\lambda}\}) \tag{5.5}$$

and thus:

$$\mathcal{I}(\{F_{i|\lambda}\}) = \mathcal{I}(\{E_{i|\lambda}\}) \tag{5.6}$$

□

5.1.1 Incompatibility monotones

With respect to the incompatibility quantifiers introduced in section 4.2, we will prove that those quantifiers satisfy the conditions to be convex incompatibility monotones.

Proposition 5.6. *The incompatibility weight $\mathcal{I}_W$ is a convex incompatibility monotone. As in definition 4.2, $\mathcal{I}_W(\{E_{i|\lambda}\})$ is the minimum μ such that the following decomposition is admissible:*

$$E_{i|\lambda} = \mu O_{i|\lambda} + (1-\mu) N_{i|\lambda} \tag{5.7}$$

where $\{N_{i|\lambda}\}$ and $\{O_{i|\lambda}\}$ are measurement sets, and $\{N_{i|\lambda}\}$ is jointly measurable.

Proof. The first requirement of definition 5.3 is trivially satisfied by $\mathcal{I}_W$, that is, if the observables $\{E_{i|\lambda}\}$ are jointly measurable, condition (5.7) is satisfied with $\mu = 0$. To show that the second requirement holds, let Λ be any quantum channel, and apply it to both sides of equation (5.7):

$$\begin{aligned}\Lambda(E_{i|\lambda}) &= \mu\Lambda(O_{i|\lambda}) + (1-\mu)\Lambda(N_{i|\lambda}) \\ \Lambda(E_{i|\lambda}) &= \mu\tilde{O}_{i|\lambda} + (1-\mu)\tilde{N}_{i|\lambda}\end{aligned} \tag{5.8}$$

By proposition 5.1, $\{\tilde{N}_{i|\lambda}\}$ is jointly measurable, therefore equation (5.8) provides a valid decomposition of $\Lambda(E_{i|\lambda})$.
Hence, we have that $\mathcal{I}_W(\{E_{i|\lambda}\}) \geq \mathcal{I}_R(\{\Lambda(E_{i|\lambda})\})$, and thus $\mathcal{I}$ is a incompatibility monotone. It remains to prove that it is convex.
Consider the two sets $\{E_{i|\lambda}\}$ and $\{F_{i|\lambda}\}$; we have:[2]

$$\begin{aligned}E_{i|\lambda} &= \mathcal{I}_E \; O^E_{i|\lambda} + [1-\mathcal{I}_E]\, N^E_{i|\lambda} \\ F_{i|\lambda} &= \mathcal{I}_F \; O^F_{i|\lambda} + [1-\mathcal{I}_F]\, N^F_{i|\lambda}\end{aligned} \tag{5.9}$$

Now, for any value of α, define:

$$l = \alpha\, \mathcal{I}_E + (1-\alpha)\, \mathcal{I}_F \tag{5.10}$$

and the measurements:

$$\begin{aligned}\tilde{O}_{i|\lambda} &= \frac{1}{l}\left[\alpha\, \mathcal{I}_E\; O^E_{i|\lambda} + (1-\alpha)\, \mathcal{I}_F\; O^F_{i|\lambda}\right] \\ \tilde{N}_{i|\lambda} &= \frac{1}{(1-l)}\left[\alpha\, (1-\mathcal{I}_E)\; N^E_{i|\lambda} + (1-\alpha)(1-\mathcal{I}_F)\; N^F_{i|\lambda}\right]\end{aligned} \tag{5.11}$$

with $\{\tilde{N}_{i|\lambda}\}$ being jointly measurable. We now find:

$$\alpha E_{i|\lambda} + (1-\alpha)F_{i|\lambda} = l\; \tilde{O}_{i|\lambda} + (1-l)\; \tilde{N}_{i|\lambda} \tag{5.12}$$

[2] To ease the notation, we defined $\mathcal{I}_E = \mathcal{I}_W(\{E_{i|\lambda}\})$ and $\mathcal{I}_F = \mathcal{I}_W(\{F_{i|\lambda}\})$

since this is a decomposition like the one in (5.7), we have that

$$I_W\left(\{\alpha E_{i|\lambda} + (1-\alpha)F_{i|\lambda}\}\right) \leq l \tag{5.13}$$

□

Proposition 5.7. *The incompatibility robustness $\mathcal{I}_R$ is a convex incompatibility monotone. As in definition 4.4, $\mathcal{I}_R(\{E_{i|\lambda}\})$ is the minimum t for which the set $\{O_{i|\lambda}\}$, defined as:*

$$O_{i|\lambda} = \frac{E_{i|\lambda} + t\ N_{i|\lambda}}{t+1} \tag{5.14}$$

where $\{N_{i|\lambda}\}$ is any compatible set, is jointly measurable.

Proof. The first requirement of definition 5.3 is trivially satisfied by $\mathcal{I}_R$, that is, if the observables $\{E_{i|\lambda}\}$ are jointly measurable, condition (5.14) is satisfied with $t = 0$. To show that the second requirement holds, let Λ be any quantum channel, and apply it to both sides of equation (5.14):

$$\begin{aligned} \Lambda(O_{i|\lambda}) &= \frac{\Lambda(E_{i|\lambda}) + t\ \Lambda(N_{i|\lambda})}{t+1} \\ \tilde{O}_{i|\lambda} &= \frac{\Lambda(E_{i|\lambda}) + t\ \tilde{N}_{i|\lambda}}{t+1} \end{aligned} \tag{5.15}$$

By proposition 5.1, $\{\tilde{N}_{i|\lambda}\}$ is jointly measurable, and for any t satisfying condition (5.14) $\{\tilde{O}_{i|\lambda}\}$ is as well.
Therefore, we have that $\mathcal{I}_R(\{E_{i|\lambda}\}) \geq \mathcal{I}_R(\{\Lambda(E_{i|\lambda})\})$, and thus $\mathcal{I}$ is a incompatibility monotone. The convexity follows from a similar argument as in Proposition 5.6. □

5.2 Incompatibility breaking channels

In a steering scenario, both entanglement and incompatibility are necessary resources for the setup to be steerable. In the next section, we present results showing how incompatibility is more easily spoiled than entanglement by the action of a channel, making it a more fragile resource than entanglement in a steering setup. Using an approach similar to [22], we give the following definitions:

Definition 5.8. We say that a channel Λ is an *incompatibility breaking channel* (IBC) if, for any set of POVMs $\{E_{a|\lambda}\}$ the set $\{\Lambda(E_{a|\lambda})\}$ is jointly measurable.

Definition 5.9. We say that a channel Λ is an *entanglement breaking channel* (EBC) if, for any state ρ the state[3] $\left(\Lambda^S \otimes \mathbb{1}\right)(\rho)$ is separable.

For finite dimensional Hilbert spaces, entanglement breaking channels can be fully characterized [23] as:

$$\Lambda(T) = \sum_a \mathrm{tr}[\rho_a T]\ F_a \quad \forall T \in S(\mathcal{H}) \tag{5.16}$$

where ρ_a are states and $\{F_a\}$ is an observable.

Theorem 5.10. *The set of entanglement breaking channels is a proper subset of incompatibility breaking channels. That is, EBC $\subset$ IBC [22].*

[3]With Λ^S we denote the Schödinger picture of the Heisenberg-picture channel Λ. See definition 1.13

Proof. We will first prove that all entanglement breaking channels are also incompatibility breaking. Take a general set of incompatible observables $\{E_{a|1...n}\}$ and an entanglement breaking channel Λ written as in (5.16). Now define:

$$G(a_1, a_2, \ldots .a_n) = \sum_a \text{tr}[\rho_a E_{a|1}] \ldots \text{tr}[\rho_a E_{a|n}] F_a \tag{5.17}$$

Using proposition 2.3 we find that G is a joint observable for $\{\Lambda(E_{a|1...n})\}$, and thus $EBC \subseteq IBC$.
To prove that EBC is a proper subset, we need to find a incompatibility breaking channel which is not entanglement breaking. Consider the family of white noise channles Λ^{wn}_{α}, defined as:

$$\Lambda^{wn}_{\alpha}(T) = \alpha\, T + (1-\alpha)\frac{\mathbb{1}}{d}\,\text{tr}[T] \tag{5.18}$$

where d is the dimension of the system. As shown in [22], this channel is entanglement breaking if and only if $t \leq \frac{1}{d+1}$ and it is incompatibility breaking for:

$$t \leq \frac{(3d-1)(d-1)^{(d-1)}}{(d+1)d^d} \tag{5.19}$$

therefore, choosing a value of t between the two limits gives an incompatibility breaking channel which is not entanglement breaking, completing the proof. □

Conclusions and Outlooks

The work was set out to explore the connections between joint measurability of generalized observables and the steerability of different scenarios, with particular focus on incompatibility as a necessary experimental resource for quantum steering. Original results on the steerability of channels have been presented, expanding the known link between joint measurability and steering to the realm of quantum channels. This perspective allowed us to realize the importance of incompatibility in experiments looking to demonstrate 'quantum' behaviour (*e.g.* the need for incompatibility of Alice's observables in steering), and prompted us to discuss a resource theory for it. In particular, the relevance of incompatibility as a resource is highlighted by the knowledge that it is more easily spoiled than entanglement, as any entanglement breaking scenario will necessarily break incompatibility, but not the opposite.

The work opens the possibility for future research directions, including quantification of channel steering and the construction of explicit local hidden operation models. Moreover, the usage of Bob's SE observables allows for the formulation of the steerability problem for channels as an SDP. Another related direction is to extend the notion of one-way steering to quantum channels.

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