INCOMPATIBILITY OF UNBIASED QUBIT OBSERVABLES AND PAULI CHANNELS

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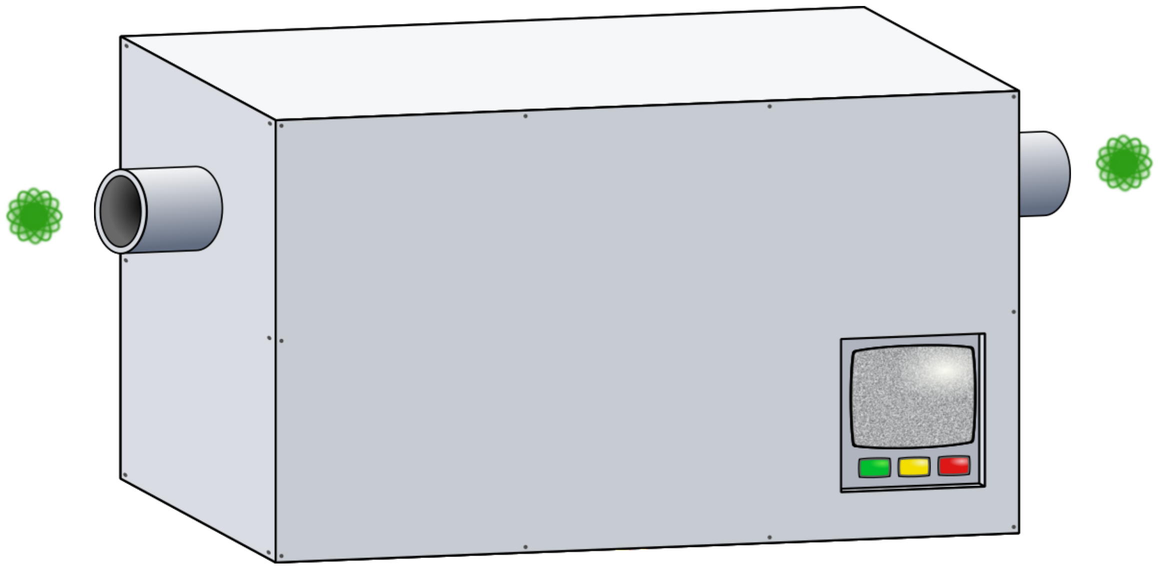
S A S P R O

What is incompatibility?

- No information without disturbance – no information = no disturbance



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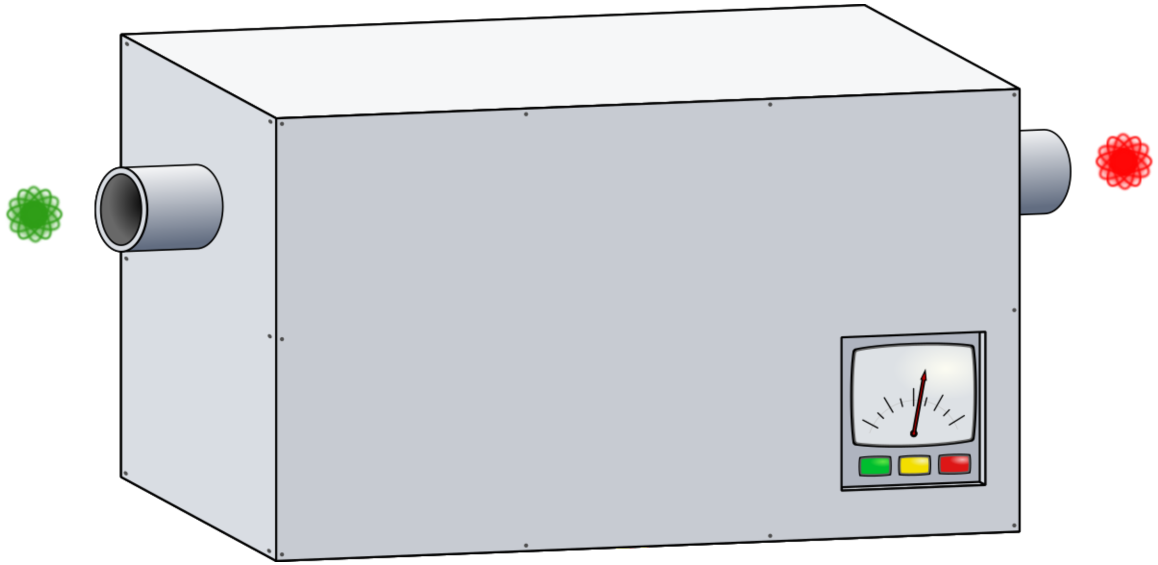
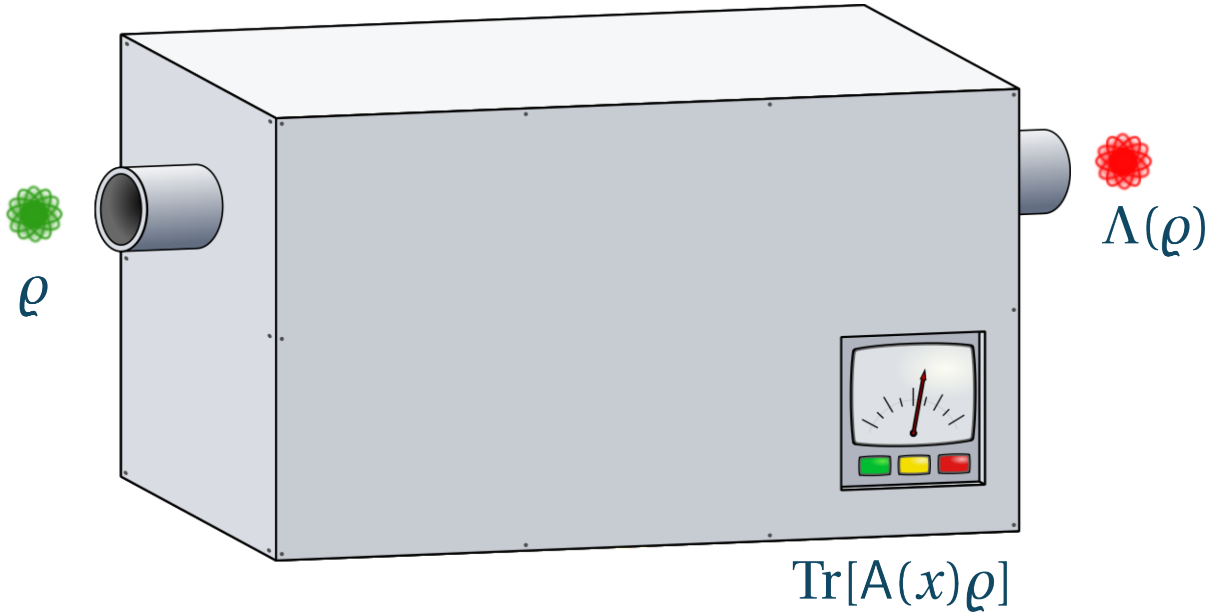


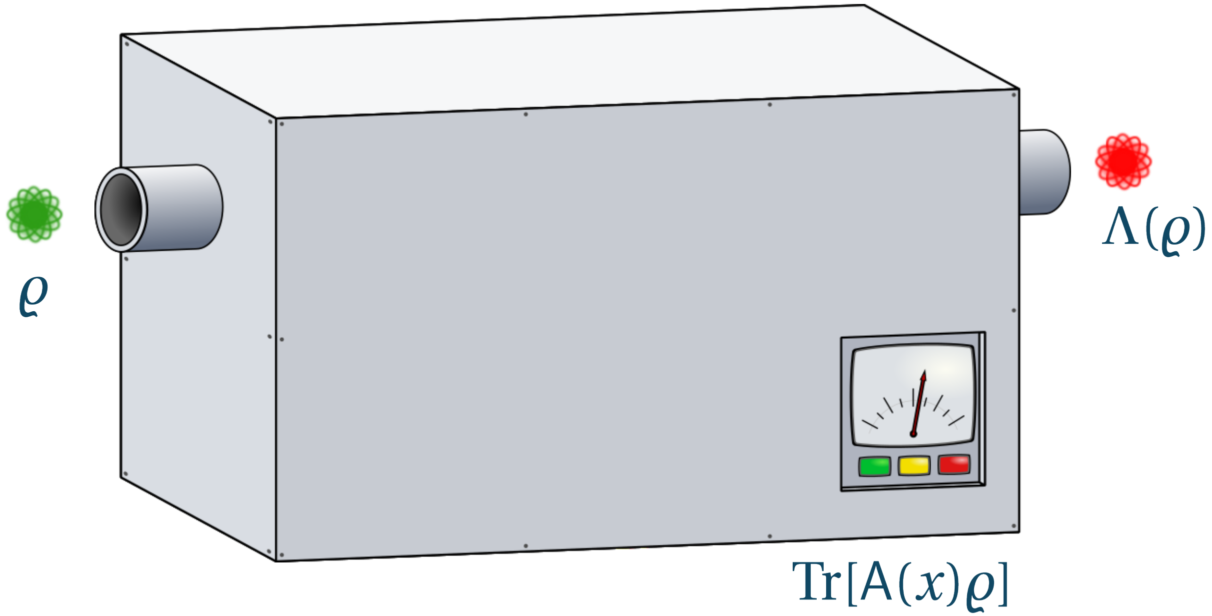
What do we want to see?

- Are channel Λ and observable A compatible? – Two points of view!

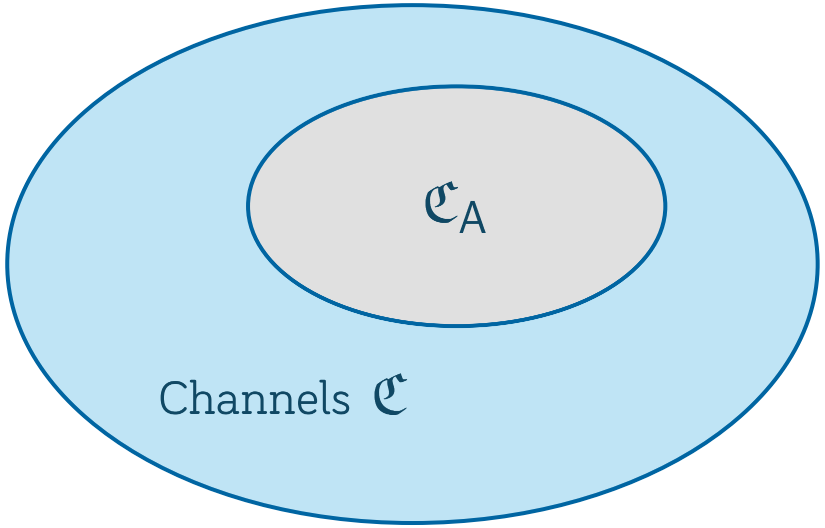


Channels compatible with an observable?

- What channels are compatible with A?



$$\mathscr{L}_{\mathsf{A}}(\varrho) = \sqrt{\mathsf{A}(+)}\varrho\sqrt{\mathsf{A}(+)} + \sqrt{\mathsf{A}(-)}\varrho\sqrt{\mathsf{A}(-)}$$



- convex
- contains Lüder's channel

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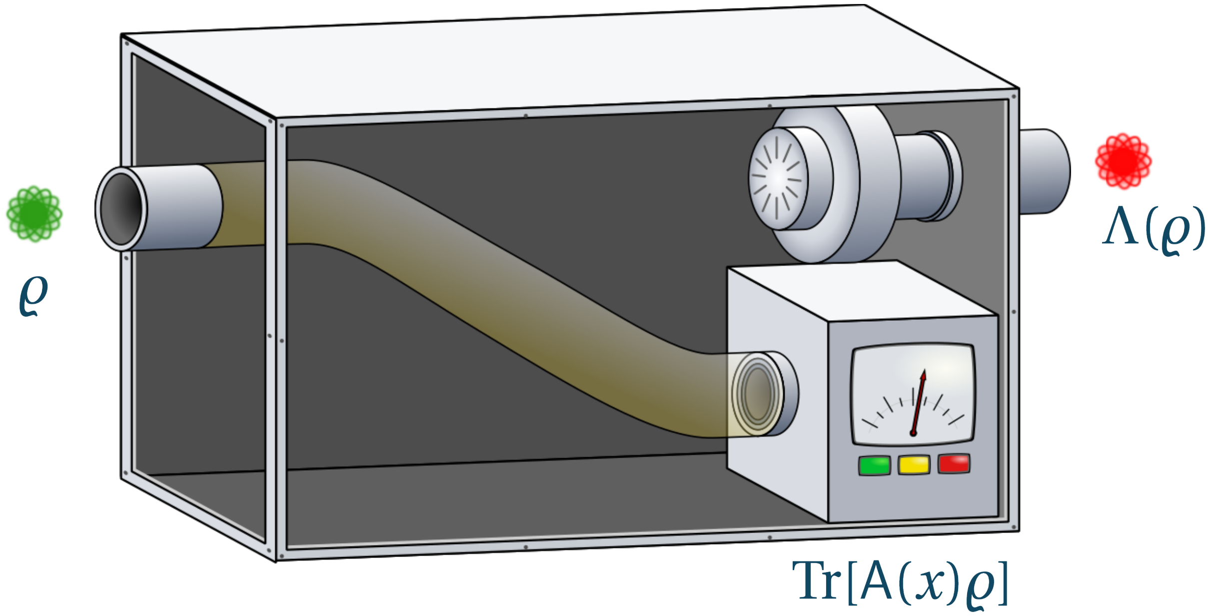

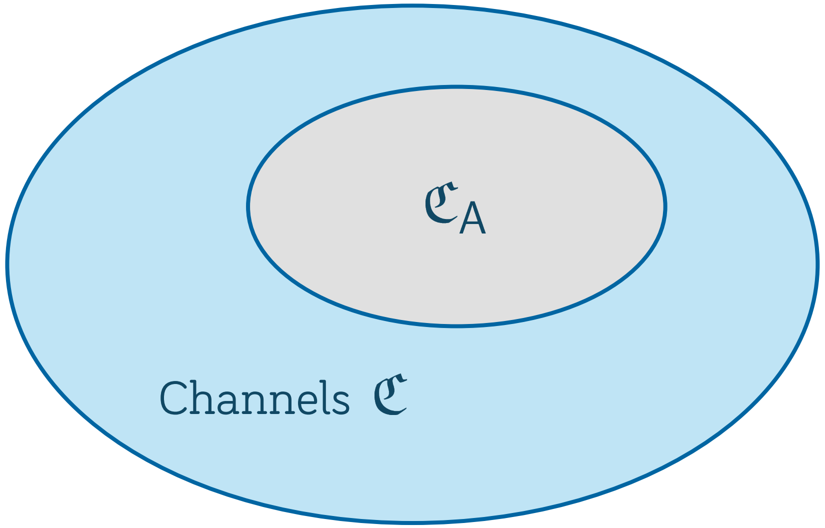

- convex
- contains Lüder’s channel
- contains all completely depolarizing channels
- left ideal of the set of all channels

Channels compatible with an observable?

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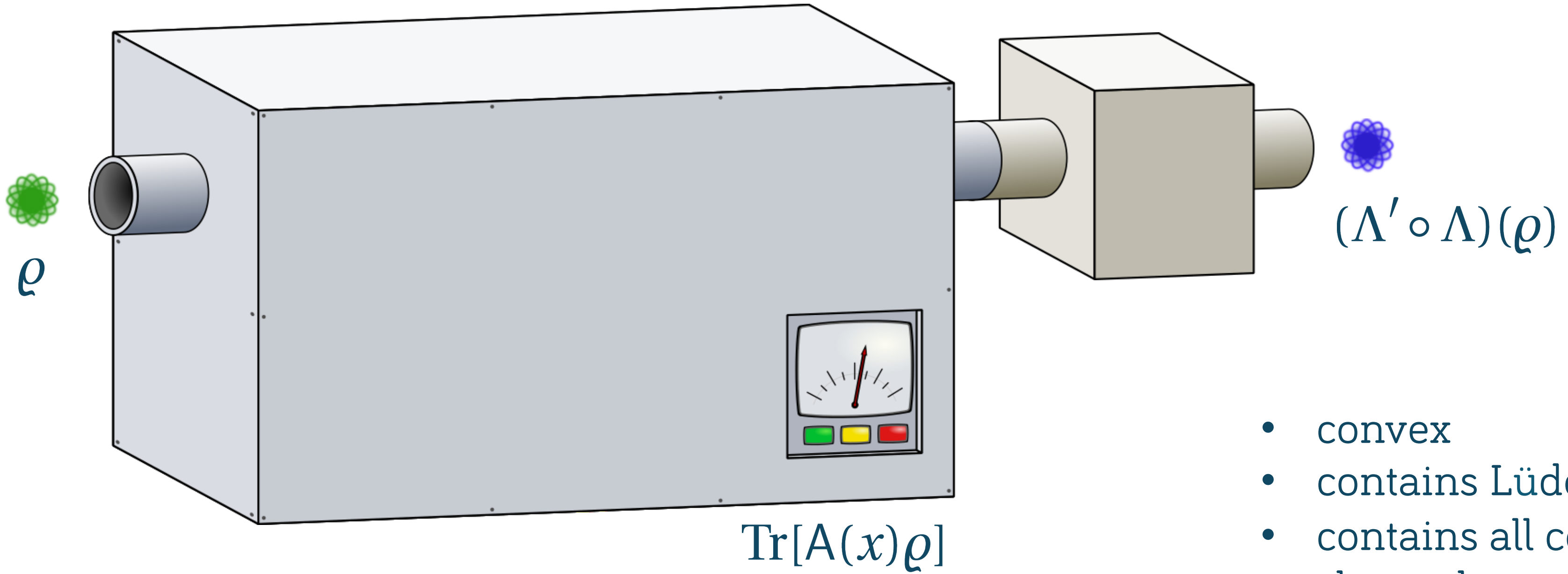
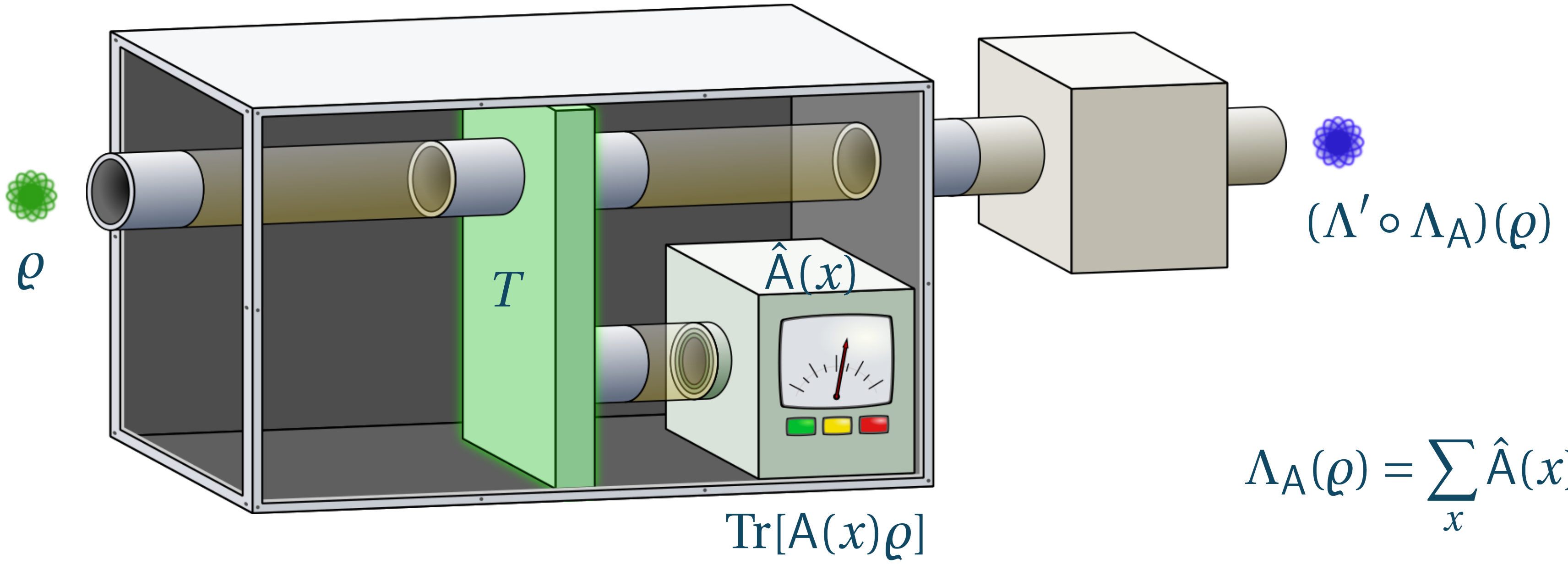


- convex
- contains Lüder's channel
- contains all completely depolarizing channels
- left ideal of the set of all channels

Channels compatible with an observable?

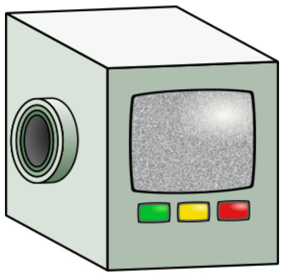
- What channels are compatible with A? – Minimal Naimark dilation of A and some subsequent channel

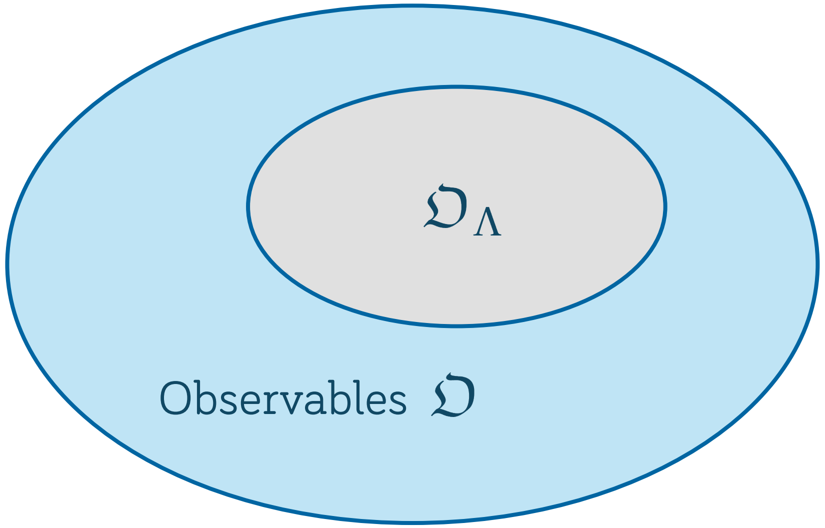


$$\Lambda_{\mathsf{A}}(\varrho) = \sum_x \hat{\mathsf{A}}(x)\, T\varrho T^* \hat{\mathsf{A}}(x)$$

Observables compatible with a channel?

- What observables are compatible with Λ?

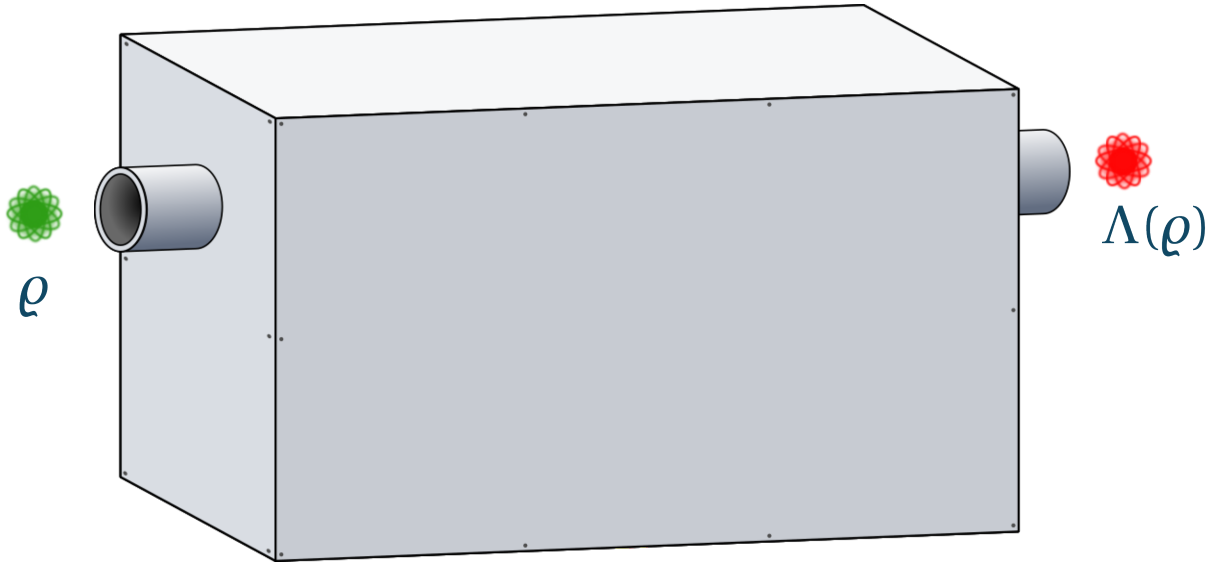




- convex
- closed under postprocessing
- contains trivial observable

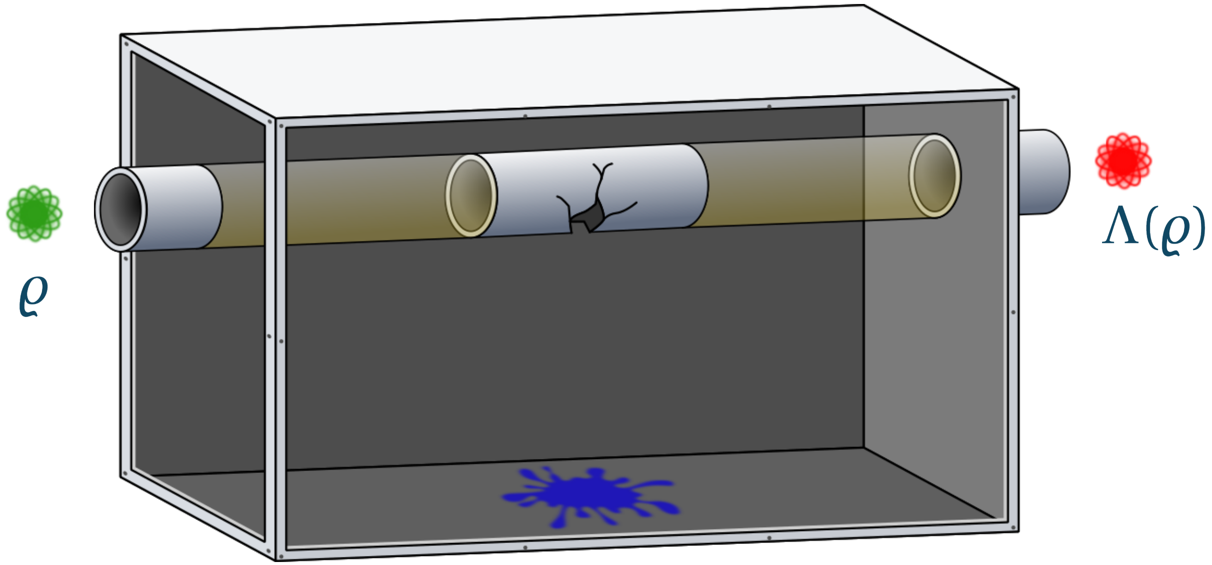
Observables compatible with a channel?

- What observables are compatible with Λ?



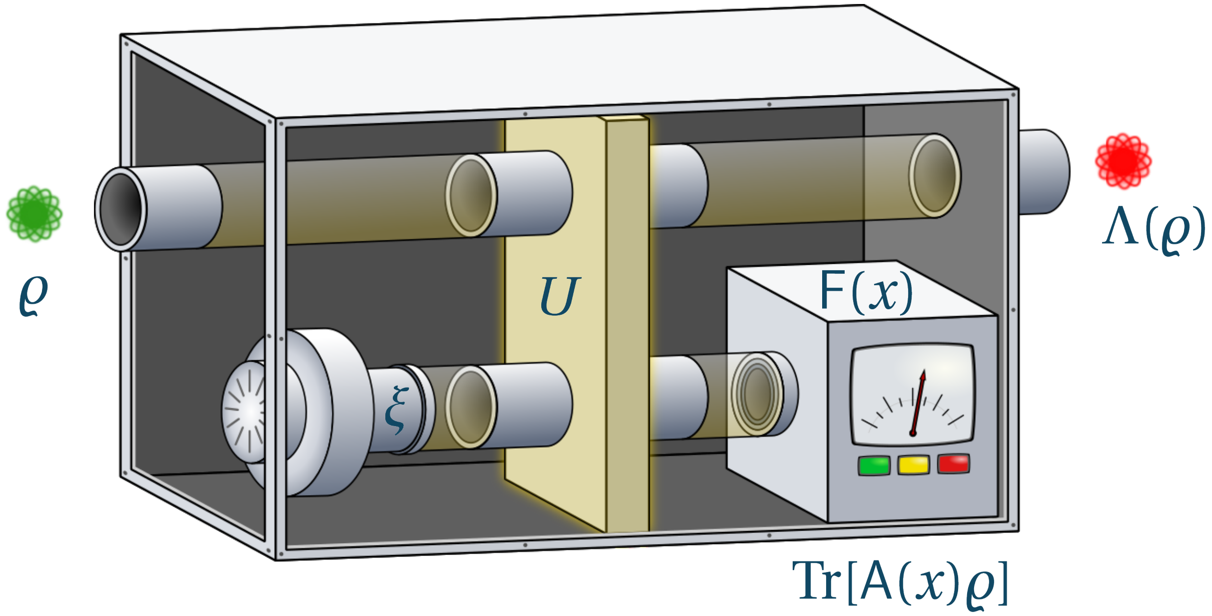
Observables compatible with a channel?

- What observables are compatible with Λ? Channel may be a consequence of some inherent losses



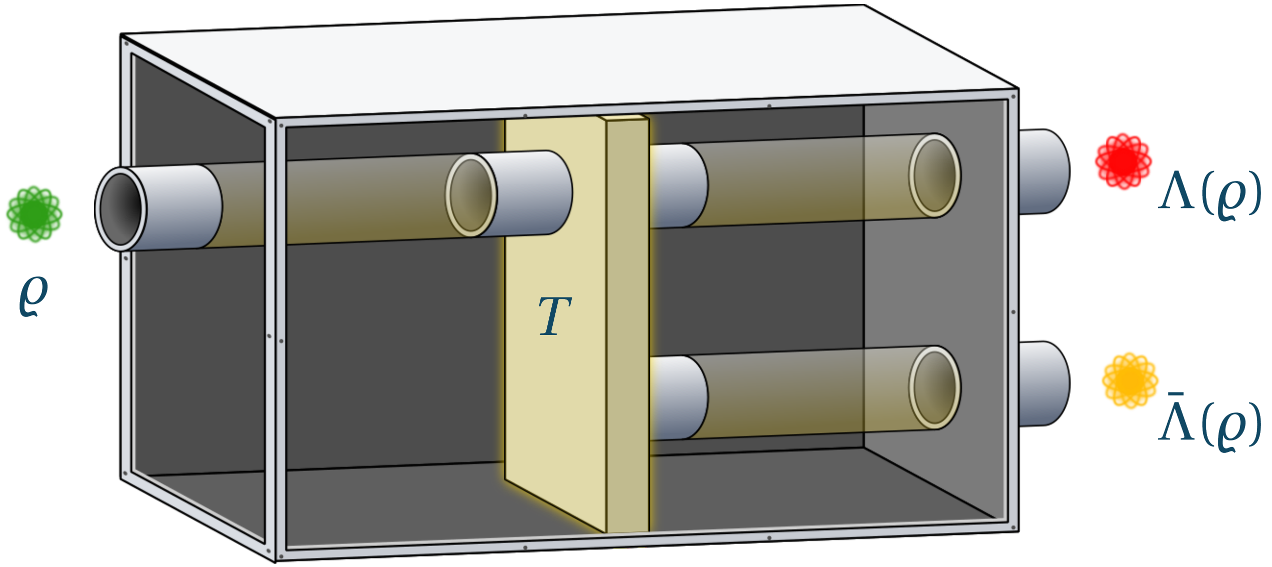
Observables compatible with a channel?

- What observables are compatible with Λ? Channel can be also a consequence of a loss of information that leaked to an eavesdropper



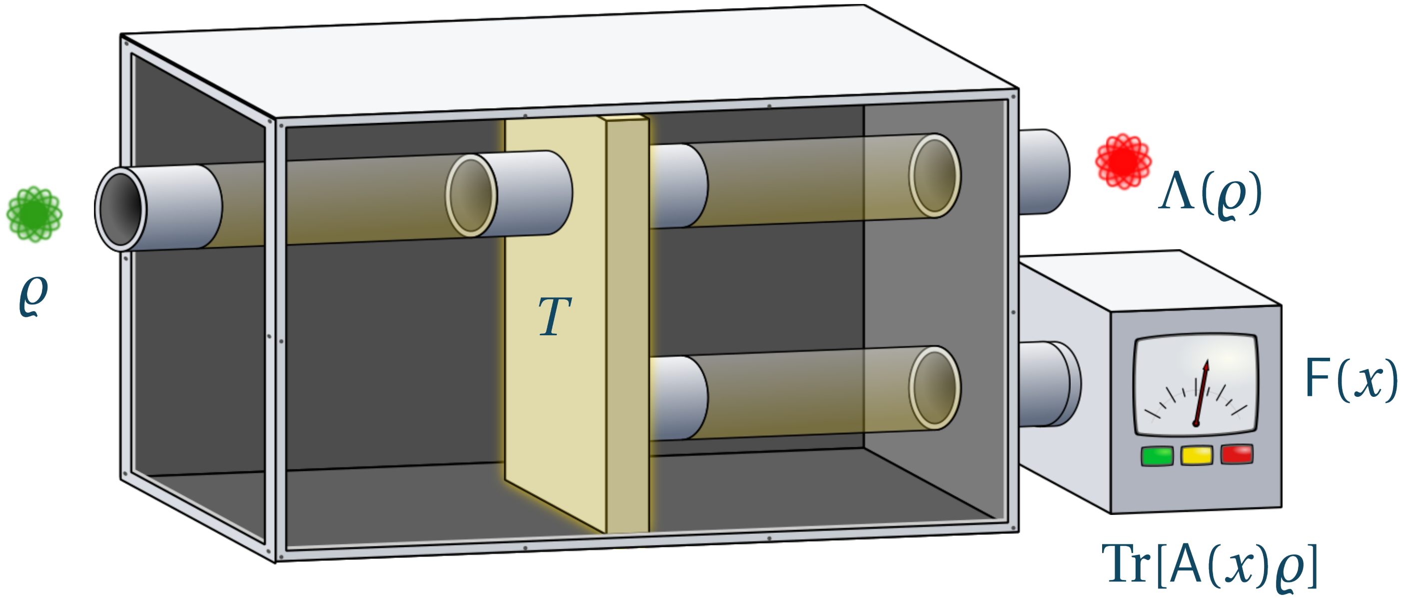
Observables compatible with a channel?

- What observables are compatible with Λ? Stinespring dilation gives us an additional *conjugate* channel.



Observables compatible with a channel?

- What observables are compatible with Λ?



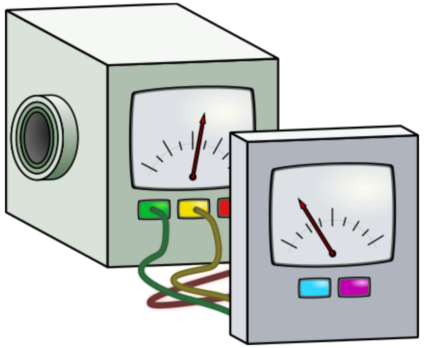
$$\mathsf{A}(x) = \bar{\Lambda}^*(\mathsf{F}(x))$$

$$\mathsf{A}(x) = \sum_{k,l} \langle e_k | \mathsf{F}(x) e_l \rangle M_k^* M_l$$

$$= \sum_{k,l} F_{kl}(x) M_k^* M_l$$

Unbiased, binary, qubit observables

- Unbiased observable gives for completely mixed state outcomes from uniform distribution
- Binary observables have two outcomes
- Qubits are nice



$$\mathrm{A}_{s,\vec{n}}(\pm) = \frac{1}{2}(\mathbb{I} \pm s\vec{n} \cdot \vec{\sigma})$$

- Postprocessing only decreases *s* – we will never get after processing more information; therefore our objective is to find maximal *s*

Unbiased, ~~binary~~, qubit channels

- Unbiased channel:
  "Just call me unital"

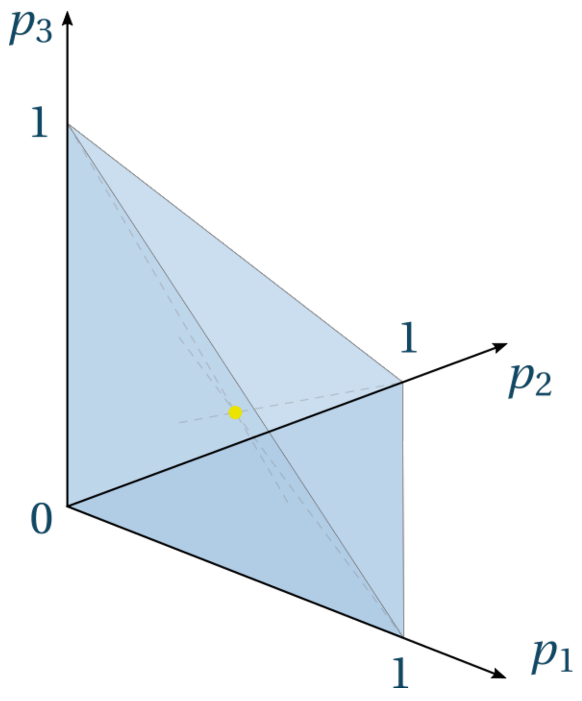
- ~~Binary channels have~~

- Qubits are nice –random unitary channels;
  we study *Pauli channels*

$$\Psi_{\vec{p}}(\varrho) = \sum_{j=0}^{3} p_j \sigma_j \varrho \sigma_j$$

characterized by probability vector

$$\vec{p} = (p_0, p_1, p_2, p_3) \qquad \sum_{j=0}^{4} p_j = 1$$



- All observables compatible with $\Psi_{\vec{p}}$ are given as $A(x) = \sum_{k,l} \sqrt{p_k p_l} F_{kl}(x) \sigma_k \sigma_l$

Sufficient condition

$$\mathsf{A}_{s,\vec{n}}(\pm) = \frac{1}{2}(\mathbb{1} \pm s\vec{n}\cdot\vec{\sigma})$$

- An unbiased binary qubit observable $\mathsf{A}_{s,\vec{n}}$ and Pauli channel $\Psi_{\vec{p}}$ are compatible if

$$\frac{s^2 n_1^2}{\pi_1^2} + \frac{s^2 n_2^2}{\pi_2^2} + \frac{s^2 n_3^2}{\pi_3^2} \leq 1$$

where

$$\pi_1 = 2(\sqrt{p_0 p_1} + \sqrt{p_2 p_3})$$
$$\pi_2 = 2(\sqrt{p_0 p_2} + \sqrt{p_1 p_3})$$
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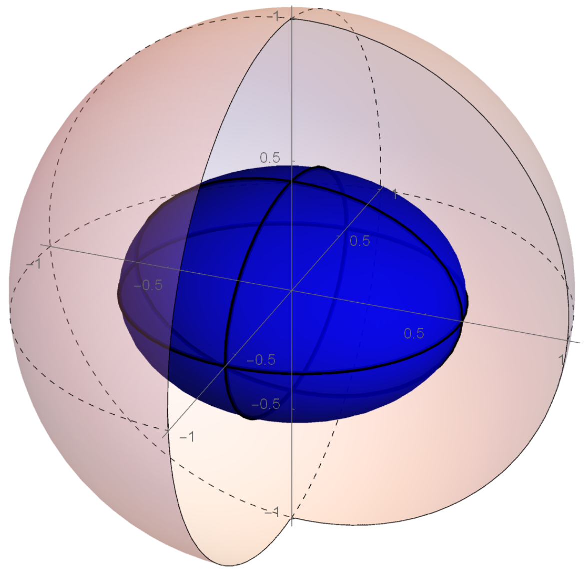

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$$F(+) \equiv F = \frac{1}{2}\begin{pmatrix} 1 & n'_1 & n'_2 & n'_3 \\ n'_1 & 1 & -in'_3 & in'_2 \\ n'_2 & in'_3 & 1 & -in'_1 \\ n'_3 & -in'_2 & in'_1 & 1 \end{pmatrix}$$

$$\vec{n}' = \frac{Q^{-1}\vec{n}}{\|Q^{-1}\vec{n}\|} \qquad Q_{ij} = \delta_{ij}\pi_j$$

- *Proof:*
  Not really elegant… we just provide $F(\pm)$, which is a projection

Necessary condition

$$\mathsf{A}_{s,\vec{n}}(\pm) = \frac{1}{2}(\mathbb{1} \pm s\vec{n}\cdot\vec{\sigma})$$

- An unbiased binary qubit observable $\mathsf{A}_{s,\vec{n}}$ and Pauli channel $\Psi_{\vec{p}}$ are compatible only if
$$\frac{s^2 n_1^2}{\pi_1^2} + \frac{s^2 n_2^2}{\pi_2^2} + \frac{s^2 n_3^2}{\pi_3^2} \le 1$$
- *Proof:*
  – NONE –

- But in specific cases we have it: A Pauli channel $\Psi_{\vec{p}}$ is compatible with $\mathsf{A}_{s,\vec{e}_j}$ if and only if $|s| \le \pi_j$.
- *Proof:*
  Showing that $\mathsf{F}(\pm)$ we have in the sufficient condition is optimal.

SDP formulation

Let

$$\Sigma_x = 2\begin{pmatrix} 0 & \sqrt{p_0 p_1} & 0 & 0 \\ \sqrt{p_0 p_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{p_2 p_3} \\ 0 & 0 & i\sqrt{p_2 p_3} & 0 \end{pmatrix}$$

$$\Sigma_y = 2\begin{pmatrix} 0 & 0 & \sqrt{p_0 p_2} & 0 \\ 0 & 0 & 0 & i\sqrt{p_1 p_3} \\ \sqrt{p_0 p_2} & 0 & 0 & 0 \\ 0 & -i\sqrt{p_1 p_3} & 0 & 0 \end{pmatrix}$$

$$\Sigma_z = 2\begin{pmatrix} 0 & 0 & 0 & \sqrt{p_0 p_3} \\ 0 & 0 & -i\sqrt{p_1 p_2} & \\ 0 & i\sqrt{p_1 p_2} & 0 & 0 \\ \sqrt{p_0 p_3} & 0 & 0 & 0 \end{pmatrix}$$

Primal problem:

$$s = \max \mathrm{Tr}[X(\vec{n}\cdot\vec{\Sigma})]$$

Subject to:

$$0 \leq X \leq \mathbb{I}$$

$$\mathrm{Tr}[X(\vec{n}_\perp\cdot\vec{\Sigma})] = 0$$

Dual problem:

$$s = \min \mathrm{Tr}[\lambda]$$

Subject to:

$$\lambda \geq (\vec{n} + \nu\vec{n}_\perp)\cdot\vec{\Sigma}$$

$$\lambda \geq 0 \qquad \nu \in \mathbb{R}$$

SDP formulation – specific cases with $\nu = 0$

Primal problem:

$$s = \max \mathrm{Tr}[X(\vec{n}\cdot\vec{\Sigma})]$$

Subject to:

$$0 \leq X \leq \mathbb{I}$$

$$\mathrm{Tr}[X(\vec{n}_\perp\cdot\vec{\Sigma})] = 0$$

Dual problem:

$$s = \min \mathrm{Tr}[\lambda]$$

Subject to:

$$\lambda \geq \vec{n}\cdot\vec{\Sigma}$$

$$\lambda \geq 0$$

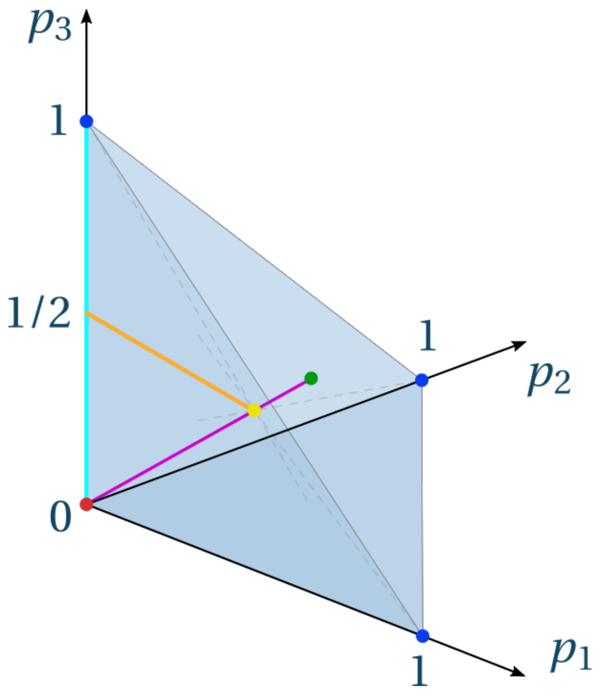
If P is the projector to the positive subspace, then

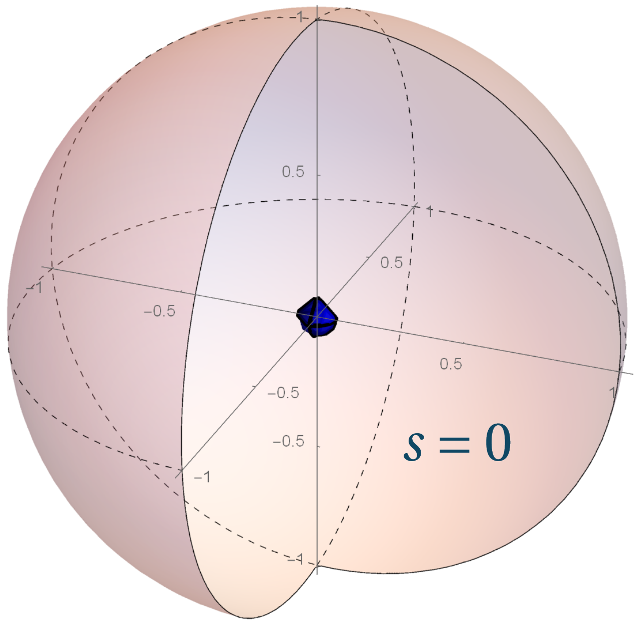
- let $\lambda = P(\vec{n}\cdot\vec{\Sigma})P$ and $X = P$ then $s = \mathrm{Tr}[P(\vec{n}\cdot\vec{\Sigma})]$, dual conditions are satisfied together with X being an effect, but in general
$$\mathrm{Tr}[P(\vec{n}_\perp\cdot\vec{\Sigma})] \neq 0$$
- or we can choose $X = F = F^2$ from the sufficient condition and take $\lambda = F(\vec{n}\cdot\vec{\Sigma})F$ then $s = \mathrm{Tr}[F(\vec{n}\cdot\vec{\Sigma})]$, primal conditions are satisfied together with the first dual condition, but in general $\lambda \ngeq 0$
- Enough (?): $\lambda = F(\vec{n} + \nu\vec{n}_\perp)\cdot\vec{\Sigma}F$

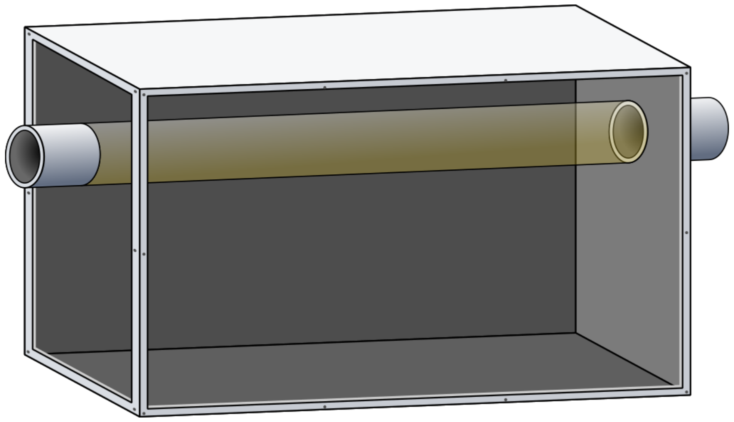
Identity channel

- Red point $\vec{p} = (1, 0, 0, 0)$

$$\Sigma_x = \Sigma_y = \Sigma_z = 0$$







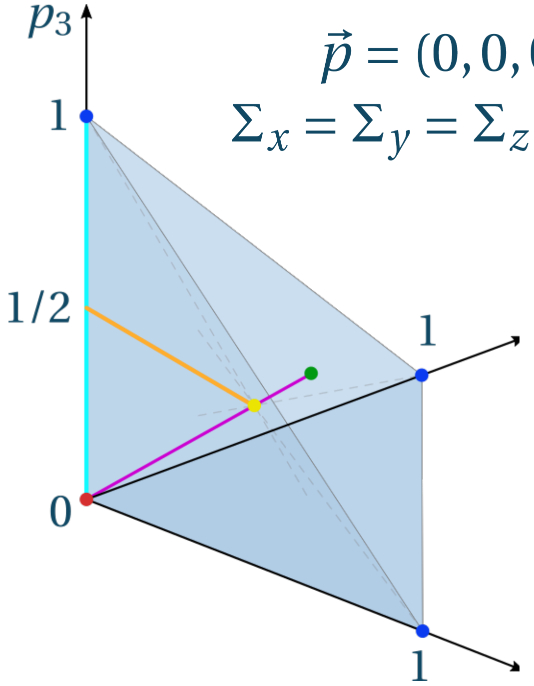
Unitary channels

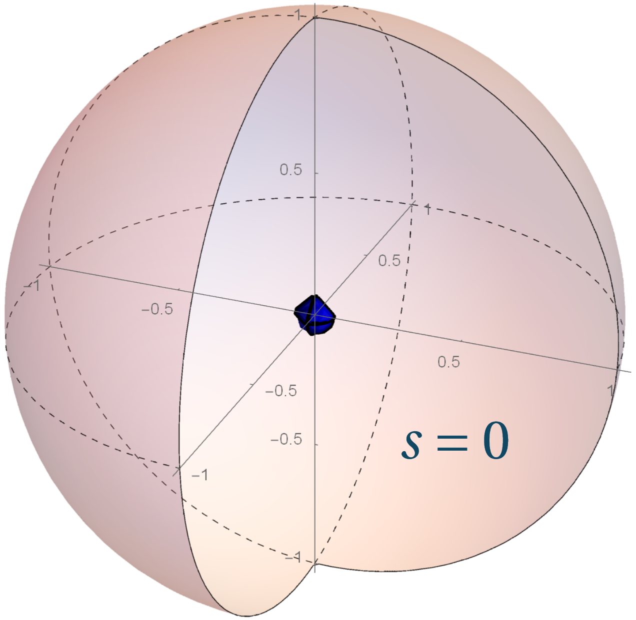
- Blue dots $\vec{p} = (0,1,0,0)$

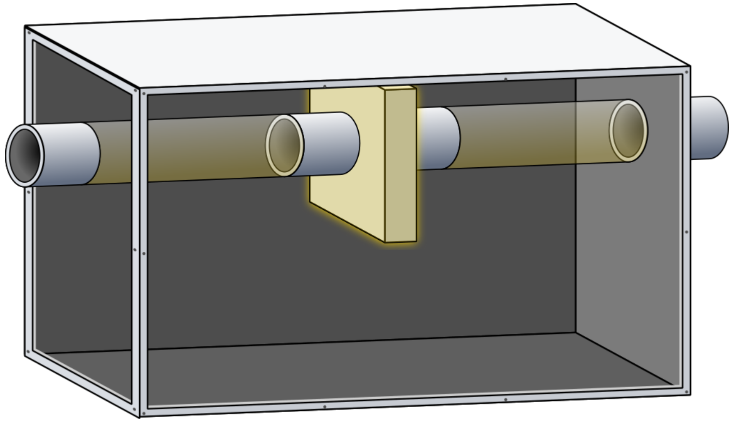
$$\vec{p} = (0,0,1,0)$$

$$\vec{p} = (0,0,0,1)$$

$$\Sigma_x = \Sigma_y = \Sigma_z = 0$$



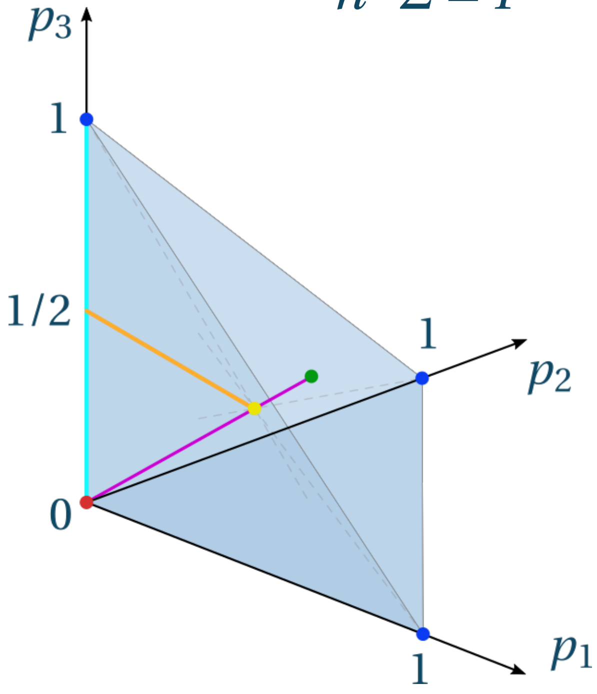


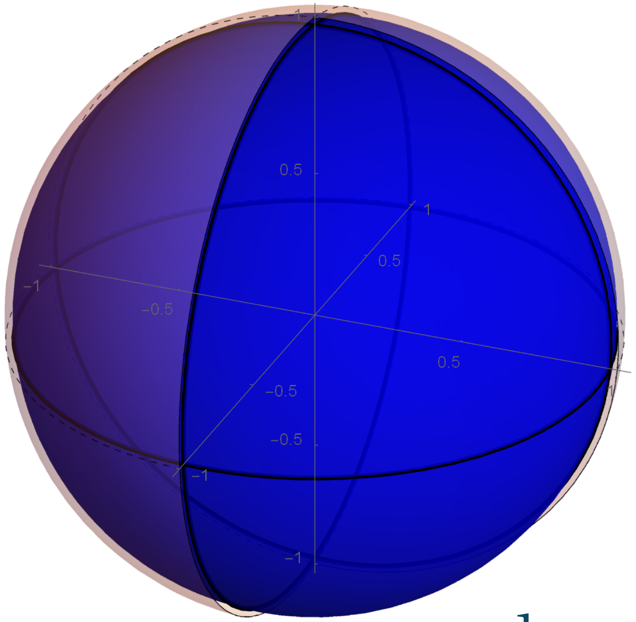


Completely depolarizing channel $\Gamma_{1/4}(\varrho) = \frac{1}{2}\mathbb{1}$

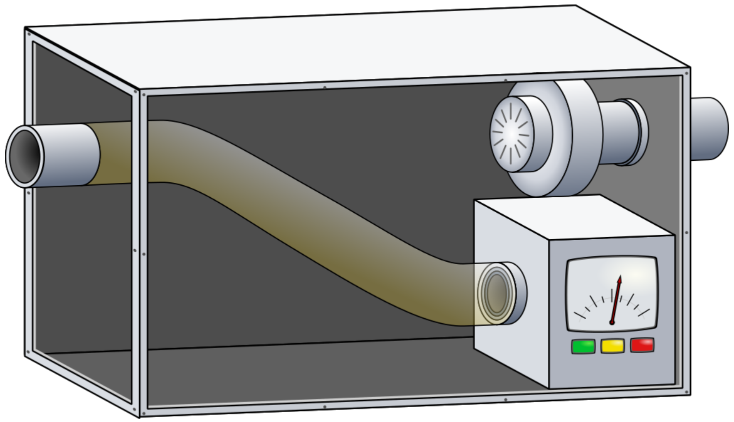
- Yellow dot $\vec{p} = \frac{1}{4}(1,1,1,1)$

$$\vec{n} \cdot \Sigma = F$$



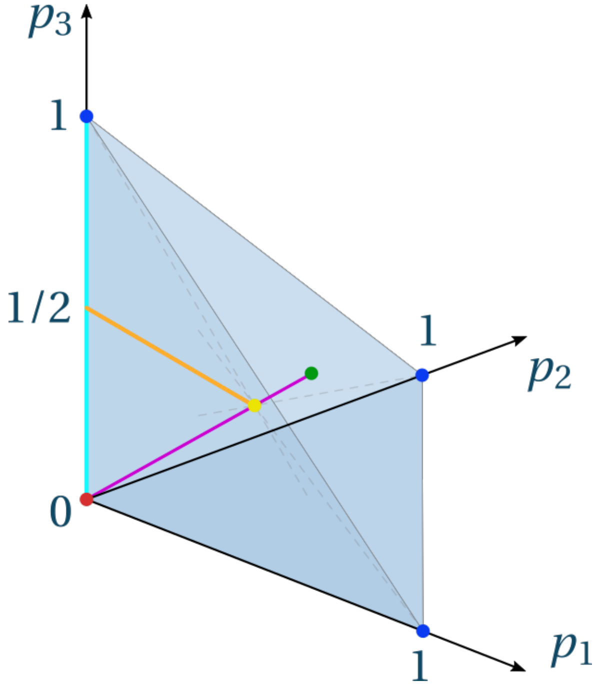


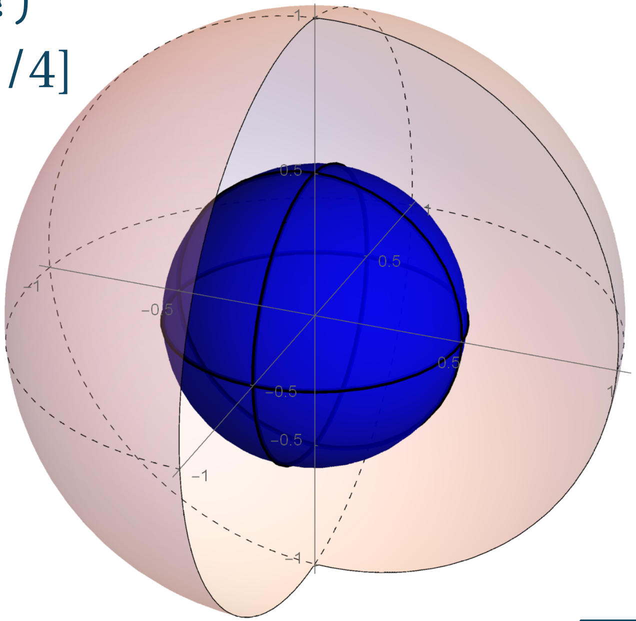
$s = 1$



Partially depolarizing channels $\Gamma_p(\varrho) = (1-4p)\varrho + 2p\mathbb{1}$

- Purple line (only sufficient?)
  $\vec{p} = (1-3p, p, p, p),\ p \in [0, 1/4]$

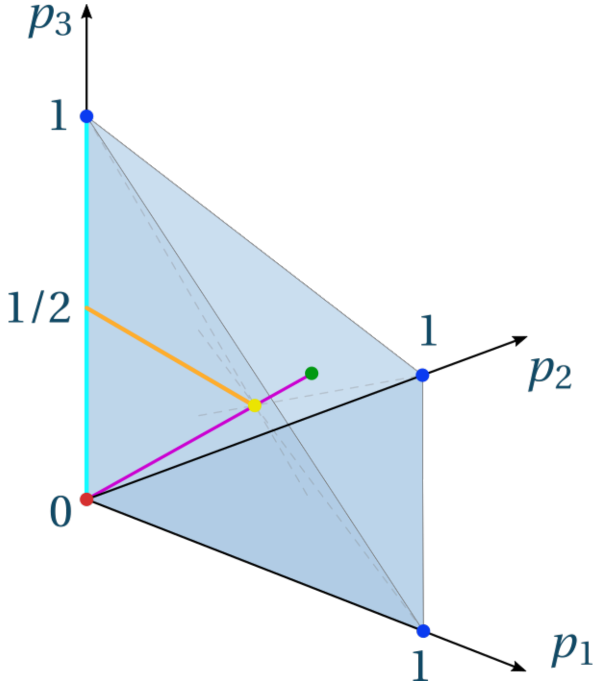


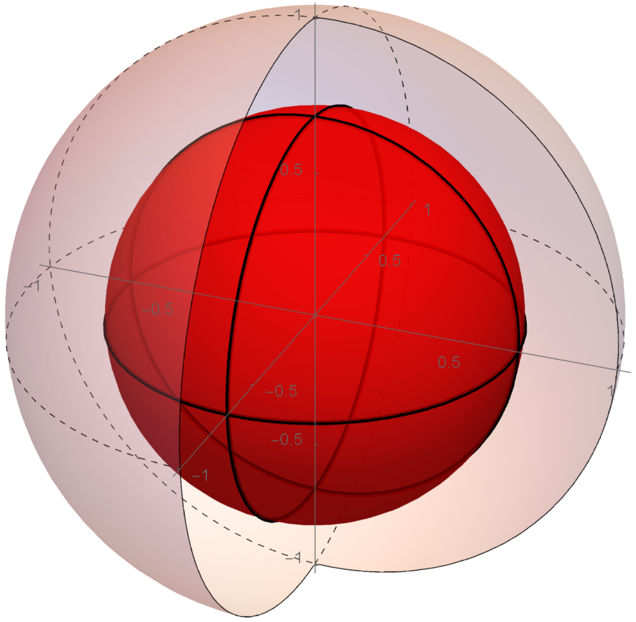


$s \leq 2[p + \sqrt{p(1-3p)}]$

Quantum NOT – $\Gamma_{1/3}$

- Green dot $\vec{p} = \frac{1}{3}(0,1,1,1)$

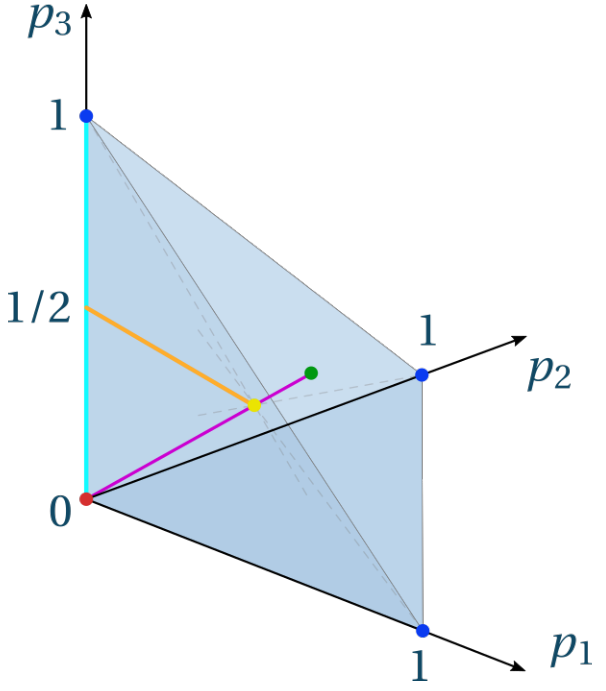


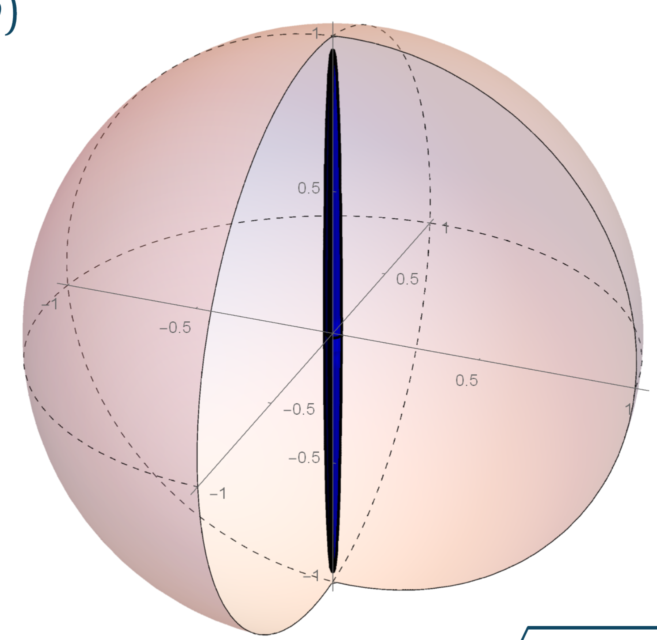


$s \leq 2/3$

Phase damping channels

- Cyan line $\vec{p} = (p, 0, 0, 1-p)$
  only one Σ_j is non-zero





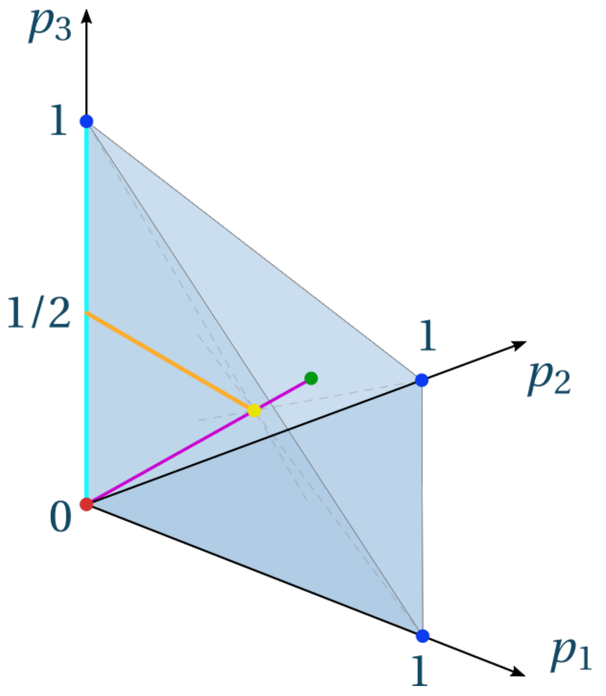
$$s \leq 2\sqrt{p(1-p)}, \qquad x = y = 0$$

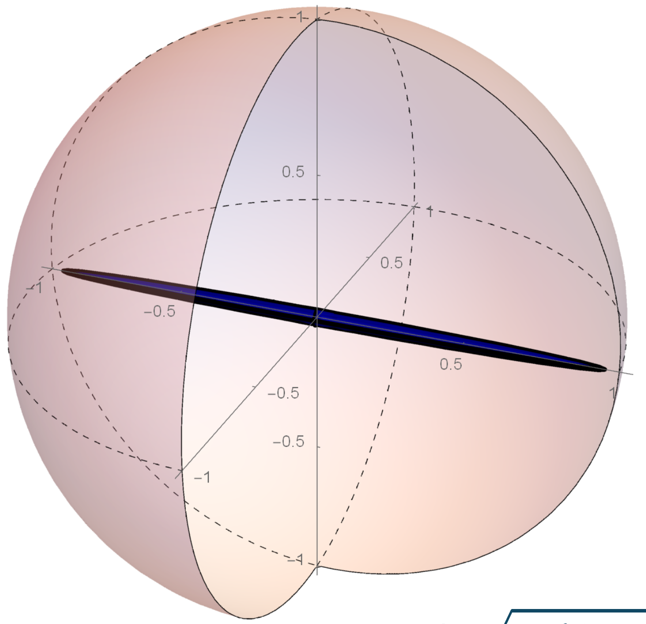
- contains Lüder's channel

$$\mathscr{L}_{\mathsf{A}}(\varrho) = \sqrt{\mathsf{A}(+)}\varrho\sqrt{\mathsf{A}(+)} + \sqrt{\mathsf{A}(-)}\varrho\sqrt{\mathsf{A}(-)}$$

Phase damping channels

- Cyan line $\vec{p} = (p, 1-p, 0, 0)$ only one Σ_j is non-zero





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Phase damping channels

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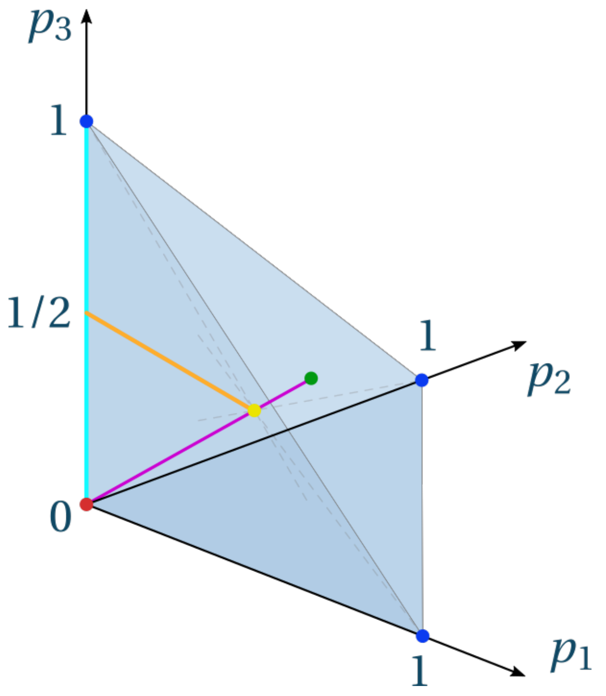
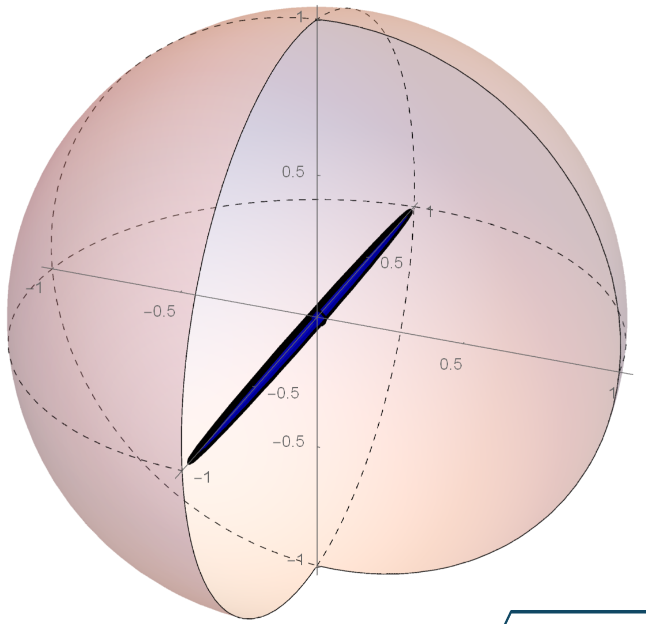




$$s \le 2\sqrt{p(1-p)}, \qquad x = z = 0$$

- contains Lüder's channel

$$\mathscr{L}_{\mathsf{A}}(\varrho) = \sqrt{\mathsf{A}(+)}\varrho\sqrt{\mathsf{A}(+)} + \sqrt{\mathsf{A}(-)}\varrho\sqrt{\mathsf{A}(-)}$$

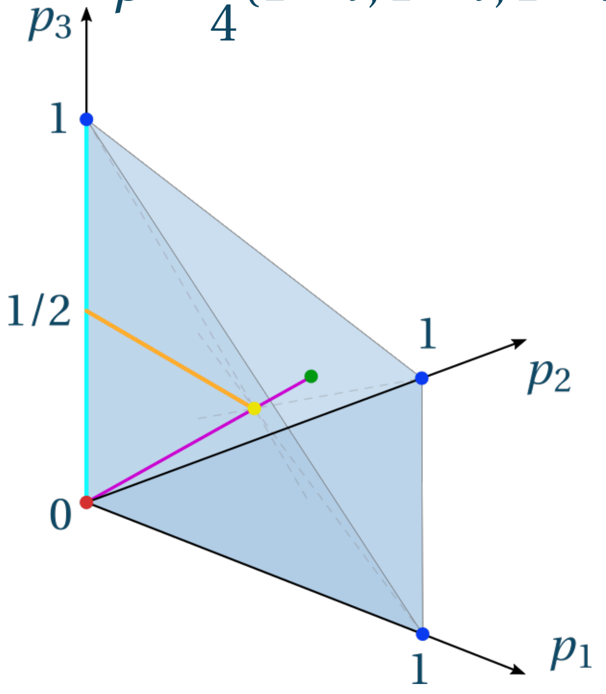
Measure and prepare channels

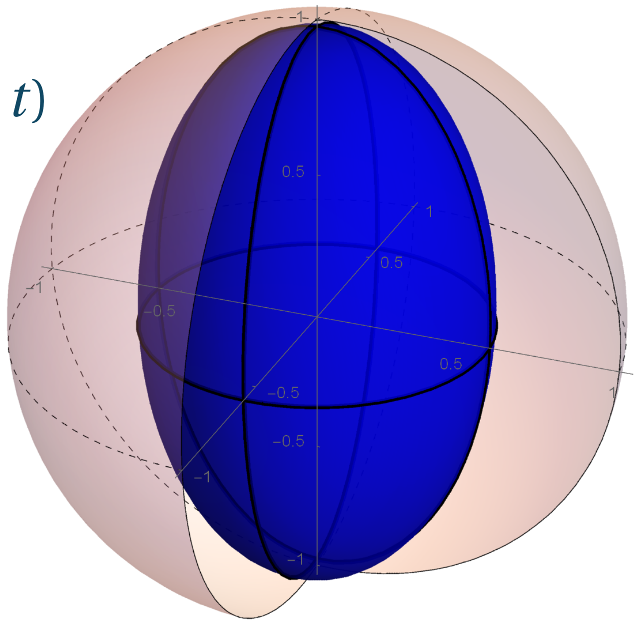
$$\mathsf{A}_{s,\vec{n}}(\pm) = \frac{1}{2}(\mathbb{I} \pm s\vec{n}\cdot\vec{\sigma})$$

$$\Theta_t(\varrho) = \mathrm{Tr}[\varrho \mathsf{A}_{t,\vec{e}_z}(+)]\mathsf{A}_{1,\vec{e}_z}(+) + \mathrm{Tr}[\varrho \mathsf{A}_{t,\vec{e}_z}(-)]\mathsf{A}_{1,\vec{e}_z}(-)$$

- Orange line

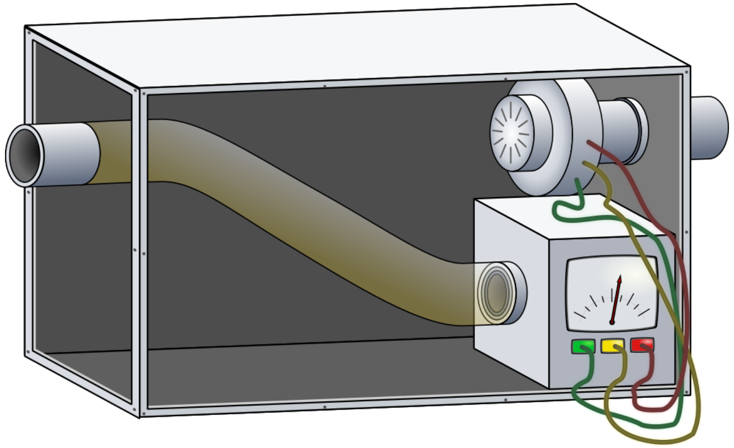
$$\vec{p} = \frac{1}{4}(1+t, 1-t, 1-t, 1+t)$$





- Composition of an absolutely damping channel and a partially depolarizing channel
- Measuring afterwards $\mathsf{A}_{s,\vec{e}_x}$,

$$s^2 + t^2 \leq 1$$



Thank you for attention!