Conditions on the compatibility of channels in General Probabilistic Theory and their connection to steering and Bell nonlocality

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The aim of the talk is to introduce operational definitions of:

- compatibility of channel
- steering by channel
- Bell nonlocality of channels

in the context of General Probabilistic Theory.

For simplicity, the talk will be limited only to quantum channels, but we will use only operational ideas.

Notation

$\mathcal{H}$, $\mathcal{K}$	finite-dimensional, complex Hilbert spaces
$B_h(\mathcal{H})$	set of self-adjoint operators on $\mathcal{H}$
$\mathfrak{D}_{\mathcal{H}}$	set of states
Tr, Tr_1, Tr_2	trace, partial trace
$C(\Phi)$	Choi matrix of channel Φ

Channel is a linear CPTP map $B_h(\mathcal{H}) \to B_h(\mathcal{K})$.

Compatibility of channels

Channels Φ_1, Φ_2 are compatible if and only if there is a channel $\Phi : B_h(\mathcal{H}) \to B_h(\mathcal{H} \otimes \mathcal{H})$ such that for every $\rho \in \mathfrak{D}_{\mathcal{H}}$ we have

$$\begin{aligned} \Phi_1(\rho) &= \mathrm{Tr}_2(\Phi(\rho)), \\ \Phi_2(\rho) &= \mathrm{Tr}_1(\Phi(\rho)). \end{aligned}$$

Note that: for any $\Phi : B_h(\mathcal{H}) \to B_h(\mathcal{H} \otimes \mathcal{H})$ we can define the channels $\mathrm{Tr}_1(\Phi)$ and $\mathrm{Tr}_2(\Phi)$ as $(\mathrm{Tr}_1(\Phi))(\rho) = \mathrm{Tr}_1(\Phi(\rho))$, $(\mathrm{Tr}_2(\Phi))(\rho) = \mathrm{Tr}_2(\Phi(\rho))$.
Hence Φ_1, Φ_2 are compatible if and only if there is a channel Φ such that

$$\begin{aligned} \Phi_1 &= \mathrm{Tr}_2(\Phi), \\ \Phi_2 &= \mathrm{Tr}_1(\Phi). \end{aligned}$$

We are "using" a canonical isomorphism between linear maps and elements of some tensor product.

Direct product of state spaces and channels

$\mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ - form of tensor product of state spaces in GPT, describes states of joint system

Definition

Let $\mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$ denote direct product of state spaces.

$\mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$ is a set of pairs of states, i.e. for $\rho_1, \rho_2 \in \mathfrak{D}_{\mathcal{H}}$ we have $(\rho_1, \rho_2) \in \mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$.
$\mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$ corresponds to conditional states, conditioned by our choice in the past. In the same way:

$$(\Phi_1, \Phi_2) : \mathfrak{D}_{\mathcal{H}} \to \mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$$

is a conditional channel. $(\Phi_1, \Phi_2)(\rho) = (\Phi_1(\rho), \Phi_2(\rho))$.

Compatibility of quantum channels, revisited

Definition

Define the map J that maps a channel $\Phi : B_h(\mathcal{H}) \to B_h(\mathcal{H} \otimes \mathcal{H})$ to a conditional channel, given as

$$J(\Phi) = (\mathrm{Tr}_2(\Phi), \mathrm{Tr}_1(\Phi)).$$

The channels Φ_1 and Φ_2 are compatible if and only if there is a channel Φ such that

$$(\Phi_1, \Phi_2) = J(\Phi).$$

We can use this condition to obtain an SDP problem for compatibility of quantum channels using their Choi matrices.

Prelude to steering and Bell nonlocality

$\rho \in \mathfrak{D}_{\mathcal{H}}$ - fixed state
If the channels Φ_1, Φ_2 are compatible, there exist a state $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ such that

$$\Phi_1(\rho) = \mathrm{Tr}_2(\sigma),$$
$$\Phi_2(\rho) = \mathrm{Tr}_1(\sigma),$$

because we can just take $\sigma = \Phi(\rho)$.

Would this be a good compatibility test?

No. For fixed $\rho \in \mathfrak{D}_{\mathcal{H}}$ we can always take

$$\sigma = \Phi_1(\rho) \otimes \Phi_2(\rho).$$

But what if ρ was an multipartite entangled state?

Some definitions

Definition

Define a map $J' : \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}} \to \mathfrak{D}_{\mathcal{H}} \times \mathfrak{D}_{\mathcal{H}}$ given for $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ as

$$J'(\sigma) = (\mathrm{Tr}_2(\sigma), \mathrm{Tr}_1(\sigma)).$$

The map J' maps bipartite states to conditional state.

Definition

Let $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ and let (Φ_1, Φ_2) be a conditional channel, then we call $(id \otimes (\Phi_1, \Phi_2))(\sigma)$ a bipartite conditional state.

Definition

Let $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ and let (Φ_1, Φ_2), (Φ_3, Φ_4) be conditional channels, then we call $((\Phi_1, \Phi_2) \otimes (\Phi_3, \Phi_4))(\sigma)$ a bipartite biconditional state.

Steering

$\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ - fixed bipartite state
If the channels Φ_1, Φ_2 are compatible, there exist a state $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$ such that

$$(id \otimes (\Phi_1, \Phi_2))(\rho) = (id \otimes J')(\sigma).$$

because we can just take $\sigma = (id \otimes \Phi)(\rho)$.

Definition

The state $\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ is steerable by the channels Φ_1, Φ_2 if we have

$$(id \otimes (\Phi_1, \Phi_2))(\rho) \notin (id \otimes J')(\mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}).$$

Results on steering

- No steering by compatible channels.
- No steering of separable states.
- For measurements, this yields the well known definition of steering.

Proposition

The maximally entangled state $|\psi^+\rangle\langle\psi^+|$ *is steerable by channels* Φ_1, Φ_2 *if and only if they are incompatible.*

Proof.

One simply obtains the Choi matrices of Φ_1, Φ_2 as the bipartite conditional state. $\square$

Results on steering

Proposition

The state $\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ *is steerable by the channels* Φ_1, Φ_2 *only if it is steerable by two copies of the identity channel* $id : \mathfrak{D}_{\mathcal{H}} \to \mathfrak{D}_{\mathcal{H}}$.

Proof.

Assume that the state $\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ is not steerable by two copies of the identity channel *id*, then there exists $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$ such that

$$\mathrm{Tr}_3(\sigma) = \rho, \qquad \mathrm{Tr}_2(\sigma) = \rho.$$

Let

$$\tilde{\sigma} = (id \otimes \Phi_1 \otimes \Phi_2)(\sigma),$$

then we have

$$\mathrm{Tr}_3(\tilde{\sigma}) = (id \otimes \Phi_1)(\rho), \qquad \mathrm{Tr}_2(\tilde{\sigma}) = (id \otimes \Phi_2)(\rho),$$

so the state ρ is not steerable by the channels Φ_1, Φ_2. □

W state example

Example

Let $\dim(\mathcal{H}) = 2$ with the standard basis $|0\rangle$, $|1\rangle$ and let

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

$|W\rangle\langle W| \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$ is the W state. We have

$$\rho_W = \mathrm{Tr}_2(|W\rangle\langle W|) = \mathrm{Tr}_3(|W\rangle\langle W|) \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}},$$

that shows that the state ρ_W is not steerable by a pair of the identity channels $id : \mathfrak{D}_{\mathcal{H}} \to \mathfrak{D}_{\mathcal{H}}$ that implies that ρ_W is not steerable by any pair of channels. Moreover it is known that the state ρ_W is entangled.

Also the only state $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$, such that $\rho_W = \mathrm{Tr}_2(\sigma) = \mathrm{Tr}_3(\sigma)$ is $\sigma = |W\rangle\langle W|$.

Bell nonlocality

$\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ - fixed bipartite state
If the channels Φ_1^1, Φ_2^1 are compatible and Φ_1^2, Φ_2^2 are compatible, there exist a state $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$ such that

$$((\Phi_1^1, \Phi_2^1) \otimes (\Phi_1^2, \Phi_2^2))(\rho) = (J' \otimes J')(\sigma).$$

because we can just take $\sigma = (\Phi^1 \otimes \Phi^2)(\rho)$.

Definition

The bipartite biconditional state $((\Phi_1^1, \Phi_2^1) \otimes (\Phi_1^2, \Phi_2^2))(\rho)$ is Bell nonlocal if

$$((\Phi_1^1, \Phi_2^1) \otimes (\Phi_1^2, \Phi_2^2))(\rho) \notin (J' \otimes J')(\mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}).$$

Result on Bell nonlocality

- No Bell nonlocality using compatible channels.
- No Bell nonlocality using separable states.
- For measurements, the definition yields the well known definition of Bell nonlocality.
- One can form a generalized version of the CHSH inequality, with maximal algebraic violation 4 and valid Tsirelson bound $2\sqrt{2}$.

Proposition

Let $\rho \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}}$ and let Φ_1^1, Φ_2^1, Φ_1^2, Φ_2^2 be channels. The bipartite biconditional state $((\Phi_1^1, \Phi_2^1) \otimes (\Phi_1^2, \Phi_2^2))(\rho)$ is Bell nonlocal only if the bipartite biconditional state $((id, id) \otimes (id, id))(\rho)$ is Bell nonlocal.

W state example

Example

Let $\rho_W = \mathrm{Tr}_2(|W\rangle\langle W|) = \mathrm{Tr}_3(|W\rangle\langle W|)$ and consider the bipartite biconditional state $((id, id) \otimes (id, id))(\rho_W)$.
It is Bell local, if there is a state $\sigma \in \mathfrak{D}_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}}$ such that

$$\mathrm{Tr}_{13}(\sigma) = \mathrm{Tr}_{14}(\sigma) = \mathrm{Tr}_{23}(\sigma) = \mathrm{Tr}_{24}(\sigma) = \rho_W.$$

$\mathrm{Tr}_3(\mathrm{Tr}_1(\sigma)) = \mathrm{Tr}_4(\mathrm{Tr}_1(\sigma)) = \rho_W$ implies that we must have

$$\mathrm{Tr}_1(\sigma) = |W\rangle\langle W|.$$

This implies that there is a state $\tau \in \mathfrak{D}_{\mathcal{H}}$ such that $\sigma = \tau \otimes |W\rangle\langle W|$. This implies that we have $\mathrm{Tr}_{23}(\sigma) = \tau \otimes \frac{1}{3}(2|0\rangle\langle 0| + |1\rangle\langle 1|)$ which is clearly a separable state. This is a contradiction as we should have had $\mathrm{Tr}_{23}(\sigma) = \rho_W$ which is an entangled state.

Open problems

- Structure of conditional states and conditional channels.
- Degrees and measures of compatibility of channels.
- Best approximation by compatible channels.
- Steering and Bell nonlocality with more than 1 fixed state.
- Lack of connection between steering and Bell nonlocality.
- Applications to cryptography.
- Resource theories of steering and Bell nonlocality.

Thank you for your attention.

For more detail see the full publication
arXiv: 1707.08650