Sequential measurements of two variables: quantum state determination, improving measurement precision, and relation to (a form of) Heisenberg uncertainty principle

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Main results

1. A measurement of position with an uncertainty ε_Q *does not* introduce a disturbance $\eta_{P|Q}$ in a subsequent momentum measurement, such that

$$\varepsilon_Q \eta_{P|Q} \geq \hbar/2.$$

This result depends on the definition of disturbance and is of fundamental interest.

2. By preceding a momentum measurement with an imprecise position measurement, it is possible to cancel the noise of the second measurement apparatus, obtaining thus an ideal momentum detector. This result does not depend on the definition of disturbance and may find applications in precision metrology.

3. Measuring position then momentum of a system allows to reconstruct its state.

4. In a finite–dimensional Hilbert space, there are families of conjugated operators A and B, such that measuring them sequential allows to reconstruct the state of a system.

Uncertainty principles

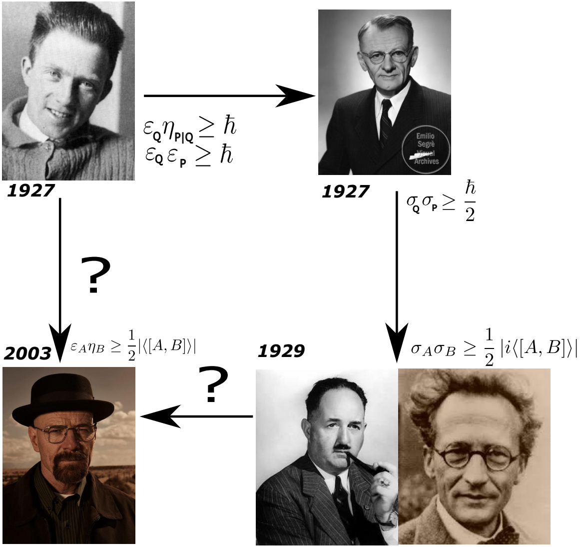


Figure: Many recent claims of violation of Heisenberg relation. Or maybe “Heisenberg’s”?

Operational definition of noise/uncertainty

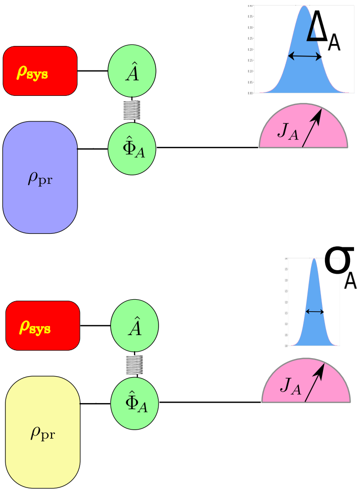


Figure: **Measure Q, with the actual apparatus and with an ideal apparatus.** Individual measurement uncertainty: $\varepsilon_Q = \sqrt{\Delta_Q^2 - \sigma_Q^2}$.

Operational definition of disturbance

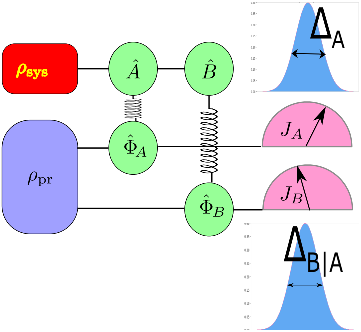


Figure: **Measure** $A = Q$ **and** $B = P$**.** Consider the variance of the pointer $\Delta^2_{P|Q}$.

Operational definition of disturbance

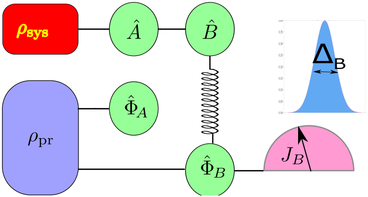


Figure: **Measure only $B = P$ with the same apparatus, by switching off the $Q\Phi_Q$ interaction.** Consider the variance of the pointer Δ_P^2.

Operational definition of disturbance

Define the squared statistical disturbance $\eta^2_{P|Q} = \Delta^2_{P|Q} - \Delta^2_P$. Notice that a priori we may have $\eta^2_{P|Q} < 0$, i.e., the measurement of Q may not disturb a subsequent measurement of P, but rather make it more precise!

Validity of Heisenberg noise–disturbance relation

Usually, it is assumed an initial state

$$|\psi_{sys}\rangle \otimes |\psi_{pr1}\rangle \otimes |\psi_{pr2}\rangle.$$

Then, you can prove that $\varepsilon_Q \eta_{P|Q} \geq \hbar/2$.

Validity of Heisenberg noise–disturbance relation

However, if one instead assumes

$$|\psi_{sys}\rangle \otimes |\psi_{pr}\rangle$$

where $|\psi_{pr}\rangle$ is an entangled state of the probes, the Heisenberg relation can be violated.

Elementary proof: operators after the measurement

Heisenberg picture. The interactions lead to the evolution

$$U = \exp[i\hat{P}\hat{\Phi}_P]\exp[i\hat{Q}\hat{\Phi}_Q]$$

for sequential measurements.

After the first interaction, the readout operator $\hat{J}_Q$ operator is shifted to

$$\hat{J}_Q + \hat{Q},$$

while the P operator is shifted to

$$\hat{P}' = \exp[i\hat{Q}\hat{\Phi}_Q]\hat{P}\exp[-i\hat{Q}\hat{\Phi}_Q] = \hat{P} + \hat{\Phi}_Q$$

After the second interaction, the pointer of the second meter becomes

$$\hat{J}''_P = \exp[i\hat{P}'\hat{\Phi}_P]\hat{J}_P\exp[-i\hat{P}'\hat{\Phi}_P] = \hat{J}_P + \hat{P}' = \hat{P} + \hat{J}_P + \hat{\Phi}_Q$$

If the first meter has a sharp distribution of $\hat{J}_Q$, $\hat{\Phi}_Q$ has a large variance (Kennard)

Elementary proof: operators after the measurement

Variance of the second pointer

$$\begin{aligned}\Delta^2_{P|Q} &\equiv \langle \hat{J}''^2_P \rangle - \langle \hat{J}''_P \rangle^2 \\ &= \langle \Big[\hat{P} + \hat{J}_P + \hat{\Phi}_Q\Big]^2 \rangle - \langle \hat{P} + \hat{J}_P + \hat{\Phi}_Q \rangle^2 \\ &= \sigma^2_P + \sigma^2_{J_P} + \sigma^2_{\Phi_Q},\end{aligned}$$

The cross-terms cancel if no initial correlations are assumed.

The disturbance is thus

$$\eta^2_{P|Q} = \sigma^2_{\Phi_Q} \geq 1/(4\sigma^2_{J_Q})$$

Uncertainty of the first measurement times disturbance of the second measurement

$$\sigma^2_{J_Q}\sigma^2_{\Phi_Q} \geq 1/4,$$

because of Kennard applied to the probe. The noise-disturbance relation holds.

Main idea

The above conclusion relied on the assumption

$$|\psi_{sys}\rangle \otimes |\psi_{pr1}\rangle \otimes |\psi_{pr2}\rangle.$$

However, why should the detectors be initially uncorrelated to each other?
We allow the more general initial state

$$|\psi_{sys}\rangle \otimes |\psi_{pr}\rangle.$$

Then an extra term appears

$$\begin{aligned}\Delta^2_{P|Q} &\equiv \langle \hat{J}''^2_P \rangle - \langle \hat{J}''_P \rangle^2 \\ &= \langle \left[\hat{J}_P + \hat{P} + \hat{\Phi}_Q \right]^2 \rangle - \langle \hat{J}_P + \hat{P} + \hat{\Phi}_Q \rangle^2 \\ &= \sigma^2_{J_P} + \sigma^2_P + \sigma^2_{\Phi_Q} + 2\kappa,\end{aligned}$$

where $\kappa = \langle \hat{J}_P \hat{\Phi}_Q \rangle - \langle \hat{J}_P \rangle \langle \hat{\Phi}_Q \rangle$
Disturbance: $\eta^2_{P|Q} = \sigma^2_{\Phi_Q} + 2\kappa$.

Question

Can $\eta^2_{P|Q} = \sigma^2_{\Phi_Q} + 2\kappa$ be negative? What does an imaginary disturbance mean?

Answer

Yes, $\eta^2_{P|Q}$ can become negative. κ can reach $-\sigma_{J_P}\sigma_{\Phi_Q}$ for perfectly anti correlated probes (EPR state).

It means that the first measurement did not disturb the second, actually it helped make it sharper.

Eu-turbance.

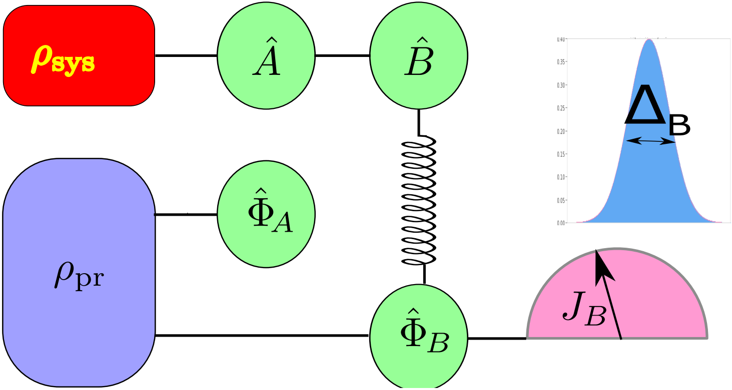
Practical consequences

Since one can reach

$$\Delta^2_{P|Q} = (\sigma_{J_P} - \sigma_{\Phi_Q})^2 + \sigma_P^2,$$

if additionally one makes sure that $\sigma_{J_P} = \sigma_{\Phi_Q}$ then the standard deviation in the P measurement is minimal.
This can be achieved by changing the coupling constants, which were absorbed for brevity $\Phi_j \to \lambda_j \Phi_j$ and by making the two variables Φ_Q, J_P perfectly anti-correlated.

A possible application



A possible application

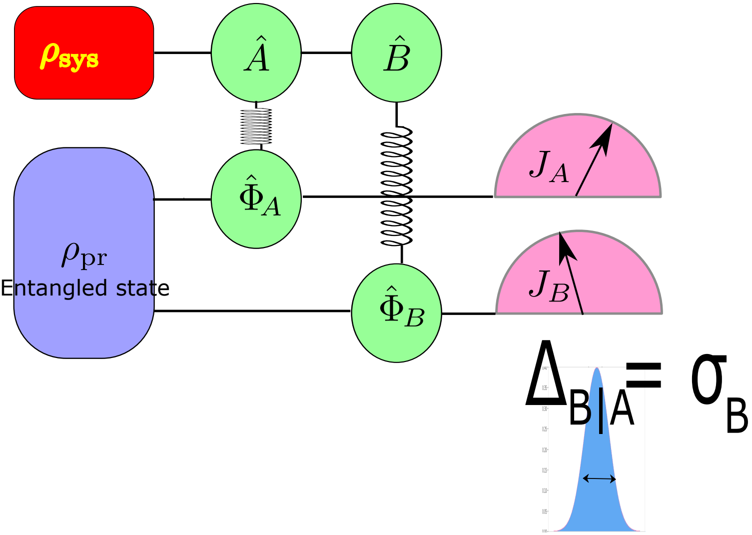


Figure: **Sequential measurements with entangled detectors.**

A possible application

Now, against common belief, it may happen that the uncertainty of the P measurement decreases compared with the ΔP when no Q measurement is made. By entangling the detectors in an EPR state, we can make $\Delta_{P|Q} = \sigma_P$, i.e., the P-detector works as an ideal detector. This results does not depend on the definition of disturbance.
ADL, PRL 120403, 2013.
Bullock & Busch, PRL 120401, 2014

Part II: quantum state reconstruction

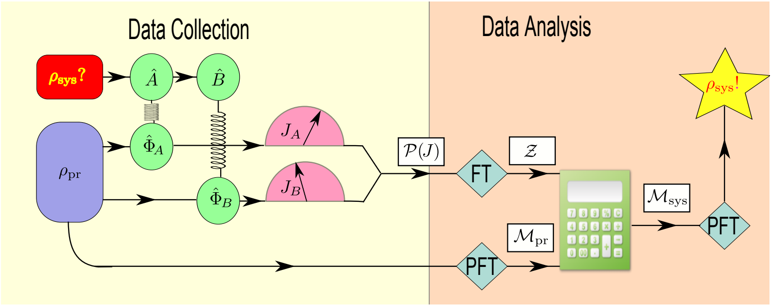


Figure: **Sequential measurements allow state reconstruction with a single setup.**

Pauli problem

Suppose you know $prob(Q) = |\psi(Q)|^2$ and $prob(P) = |\tilde{\psi}(P)|^2$.
Can you reconstruct ψ?
No

Sequential Pauli problem

Suppose you know $prob(Q, P)$ the probability of observing Q and P (or better, the corresponding pointers) in a joint or sequential measurement.
Can you reconstruct ψ?
Yes, if the measurements are not strong (projective) nor weak.

Key ingredient

Moyal quantum characteristic function

$$\mathcal{M}(\chi_p, \chi_q) = \int dPdQW(Q,P)e^{iQ\chi_Q + iP\chi_P} = \langle \exp[i\chi_p\hat{p} + i\chi_q\hat{q}] \rangle$$

Quantum state reconstruction

Fourier–transform the experimental data
$prob(Q, P) \to Z(\chi_Q, \chi_P)$.
The relation holds (if first one measures Q then P)

$$Z(\chi_Q, \chi_P) = \mathcal{M}_{pr}(\chi_P, 0; \chi_Q, \chi_P)\mathcal{M}_{sys}(\chi_Q, \chi_P)$$

Thus one can find $\mathcal{M}_{sys}$ and from it ρ_{sys}, provided that $\mathcal{M}_{pr} \neq 0$.
For a strong interaction $\mathcal{M}_{pr} = 0$ for many values of the arguments. Intuitively, if the first measurement is projective, all the information about ρ_{sys} gets lost.

Finite–dimensional case

The procedure works if A and B are conjugated variables.
In what sense, since $[A, B] = i$ cannot hold?
This question is related to the generalization of the Wigner function.
Say A has eigenvectors $|-S\rangle, |-S+1\rangle, \dots, |S\rangle$, with $d = 2S + 1$.
Let B be defined as the Hermitean operator that generates the cyclic translations of the eigenstates of A, $e^{-iB}|m\rangle = |m+1\rangle$ if $m < S$, and $e^{-iB}|S\rangle = (-1)^{d-1}|-S\rangle$.
The eigenstates of B form a mutually unbiased basis wrt the eigenstates of A,

$$|\tilde{m}\rangle = \sum_{n=-S}^{S} \frac{1}{\sqrt{d}} e^{2\pi i m n/d} |n\rangle$$

Moyal function

Given a conjugate pair A and B,

$$\mathcal{M}_{\rm sys}(\phi_A, a) = \sum_{\bar{A}} e^{i\phi_A \bar{A}} \langle \bar{A} + \frac{a}{2} | \rho_{\rm sys} | \bar{A} - \frac{a}{2} \rangle .$$

Here a ranges in $[1-d, d-1]$, d dimension of the Hilbert space. In order for this formula to be invertible it suffices to evaluate at $\phi = 2\pi m/(d-|a|)$, $m \in [(1-d+|a|)/2, (d-1-|a|)/2]$.

Equations for the finite–dimensional case

$$
\begin{aligned}
Z(\chi) =& \mathcal{M}_{\rm pr}(\chi; -\chi\sigma_+)\mathcal{M}_{\rm sys}(\chi) + \mathcal{M}_{\rm pr}(\chi; -\bar{\chi}\sigma_+)\mathcal{M}_{\rm sys}(\bar{\chi}),\\
Z(\bar{\chi}) =& \mathcal{M}_{\rm pr}(\bar{\chi}; -\bar{\chi}\sigma_+)\mathcal{M}_{\rm sys}(\bar{\chi}) + \mathcal{M}_{\rm pr}(\bar{\chi}; -\chi\sigma_+)\mathcal{M}_{\rm sys}(\chi),
\end{aligned}
$$

Conclusions

Noise-disturbance principle:
holds for uncorrelated detectors;
may not hold otherwise.

Perspectives
Finite–dimensional Hilbert spaces?

Quantum state determination:
possible for infinite and finite–dimensional Hilbert spaces;
no need of weak measurement approximation;
single setup.

Perspectives
Detectors with a discrete spectrum?
Efficiency wrt conventional tomographic scheme (Monte Carlo simulations)?