Quantum process tomography from incomplete data

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- process tomography
- “unphysical” results and maximum likelihood
- possible sources of “unphysicality” (speculations)
- incomplete data and MaxEnt principle
- MaxEnt for process tomography
Simplified experimental setup

preparator

quantum channel

measurement

IDENTIFICATION

PROBLEM

statistics
Process identification – inverse problem

- quantum theory \( Q[(\mathcal{E}, \rho_j, F_k)] = p_{jk} \)
- preparator identification \( \rho \)
- application of the channel \( \rho' = \mathcal{E} \otimes \mathcal{F}[\rho] \)
- measurement POVM \( \{F_k\} \quad p_k(\rho) = \text{Tr}\rho'F_k \)
- processing of exp. data to learn \( \mathcal{E} \)
  - inverse process reconstruction \( \mathcal{R}_{\text{inv}}[(\rho_j, F_k, p_{jk}] = \mathcal{E} \)
  - statistical process estimation \( \mathcal{R}[(\rho_j, F_k, \text{data})] = \mathcal{E} \)
Process identification difficulties

- inverse process reconstruction \( R_{\text{inv}}((\rho_j, F_k, p_{jk}) = E \)
  - system of linear equations
  - easy to implement
  - "unphysical" results

- statistical process estimation \( R[(\rho_j, F_k, \text{data})] = E \)
  - maximum likelihood methods
  - physical result is guaranteed
  - difficult optimization problem (exp)
Unphysical results

- problems with probabilities (small statistics)
- problems with experimental setup
  - preparator (tomography, uncorrelated)
  - measurement device (calibration)
  - noise can be included in description
  - memory effects
- imperfect preparators vs “artificial” unphysically
Model of open system dynamics

- completely positive tracepreserving linear maps
- addition of fixed uncorrelated ancilla
- unitary system dynamics of system+ancilla
- discarding the ancilla
- composition of three maps \( \mathcal{E} = \mathcal{T}_{\text{anc}} \circ \mathcal{U} \circ \mathcal{P} \)
- preparation map \( \mathcal{P} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_{\text{anc}} \)
- potential source of “unphysicality”
- CP maps \( \Leftrightarrow \mathcal{P} [\rho] = \rho \otimes \xi_{\text{fixed}} \)
- ?verification? of preparators
Accessible transformations

- all those for which the ancilliary model exists

characterization:
- arbitrary mapping, \( f : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}) \), i.e.
  \[ \rho \rightarrow \rho' = f(\rho) \]
- implementation \( \mathcal{P}[\rho] = \rho \otimes f(\rho) \)
- SWAP gate \( \rho' = \text{Tr}_{\text{anc}}[U_{\text{SWAP}} \rho \otimes f(\rho) U_{\text{SWAP}}] = f(\rho) \)

very artificial construction

is it a bad news?

linear accessible transformations
Universal NOT in a lab

- experimental situation:
  - black box and qubits (let's say spins)
  - preparation: SG measurement
  - observation: outputs orthogonal to inputs

- is it unphysical?
  - given qubits are entangled to qubits in the black box (singlet)
  - interaction via SWAP gate
Preparator devices

- independence of preparators and channels
- insight into physics behind the preparation
- E1:
  - preparator of ground state
  - all pure states prepared via unitary processing
- E2: intermediate dynamical map \( \mathcal{E}_{t_1,t_2} = \mathcal{E}_{t_2} \circ \mathcal{E}_{t_1}^{-1} \)
  - linear trace and hermiticity preserving + ???
  - NOT cannot be realized within this model
Complete process tomography

- for experiments only the **linearity** is important
- number of parameters $= d^2(d^2 - 1)$
  - exponential in number of qubits
- based on (incomplete) state tomography
- test states = lin. independent states $\{\rho_1, \ldots, \rho_{d^2}\}$
  - $d \times d$ state reconstructions $\rho_j \rightarrow \rho'_j = \mathcal{E}[\rho_j]$
- ancilla-assisted tomography
  - ancilla reduces the number of test states
  - single test state $\Omega_\mathcal{E} = \mathcal{E} \otimes I[\Psi_{ME}] = \frac{1}{d} \sum_{jk} \mathcal{E}[e_{jk}] \otimes e_{jk}$
Ancilla-assisted tomography

- ancilla-assisted test state
  \[
  \Omega_{\mathcal{E}} = \mathcal{E} \otimes I[\Psi_{ME}] = \frac{1}{d} \sum_{jk} \mathcal{E}[e_{jk}] \otimes e_{jk}
  \]
  \[e_{jk} = |j\rangle \langle k|\]

- state reconstruction of \( \Omega_{\mathcal{E}} \rightarrow \text{process} \)
  \[
  \mathcal{E} [\rho] = \text{Tr}_2[(I \otimes \rho^T) \Omega_{\mathcal{E}}]
  \]

- faithful (admissible) states
  \[
  \Omega = \sum \omega_{\mu\nu} S_\mu \otimes S_\nu
  \]
  \[
  \Omega_{\mathcal{E}} = \sum \omega_{\mu\nu} \mathcal{E}[S_\mu] \otimes S_\nu = \sum \omega_{\mu\nu}' S_\mu \otimes S_\nu
  \]

- process tomography
  \[
  [\omega'] = [\mathcal{E}].[\omega] \Rightarrow [\mathcal{E}] = [\omega'].[\omega]^{-1}
  \]
  \[
  \mathcal{E}[S_\mu] = \sum \mathcal{E}_{\mu\nu} S_\nu
  \]
  \[
  \mathcal{E}_{\mu\nu} = \text{Tr} S^\dagger_\mu \mathcal{E}[S_\nu]
  \]

Qubit process tomography

- phase damping \[ E_{ph.damp}^{(\lambda)} : \vec{r} \rightarrow \vec{r}' = (\lambda x, \lambda y, \lambda z) \]

- qubit=ion_{171}Yb^+ (Ch. Wunderlich)
Incomplete knowledge on outcome states

- exp.data do not determine all parameters
- observation level \( O = \{ \Lambda_1, \ldots, \Lambda_K \} \) (\( K < d^2 - 1 \))
  - probabilities/mean values \( r_k = \text{Tr} \rho \Lambda_k = \langle \Lambda_k \rangle_\rho \)
- unknown mean values
- what is the state? \( \rho = \frac{1}{d} (I + \vec{r} \cdot \vec{\Lambda}) \)
  - no unique answer
  - needs some extra assumption/postulate
- zero observation level
  - each state equally probable \( \Rightarrow \) av. state \( \Rightarrow \rho = \frac{1}{d} I \)
- naïve strategy ... set unknown parameters to 0
Maximum entropy principle

- E.T. Jaynes

Instead of asking what is the state we should ask what state best describes the state of our knowledge about the physical situation.

- average ↔ entropy $S(\rho) = -\text{Tr}\rho \log \rho$

- MaxEnt = choose the state with maximal entropy given the observation level constraints

$$\rho = \arg \max_{\rho} \{ S(\rho) | \langle \Lambda_j \rangle_\rho = r_j, \forall j = 1, \ldots, K \}$$
What is the channel?
- average channel $\mathcal{A}$
  - problem with measure
  - unital, because $E_{\pm} : \vec{r} \rightarrow \vec{r}' = T\vec{r} \pm i$ are CP maps
  - $U$ symmetry $\Rightarrow \mathcal{A}_\mu = \mu I + (1 - \mu)\mathcal{A}$ with $\mathcal{A}[\rho] = \frac{1}{d}I$ $\forall \rho$
  - contraction to total mixture
- no concept of channel entropy
  - capacity, minimal output entropy, distance,
  - Jamiolkowski isomorphism (ancilla-assisted PT)
Incomplete PT: naïve approach

- Transform states into total mixture
- Analysis done for single qubit channels
  - No problem for single test state (no ancilla)
  - Two/three test states numerically
- MaxEnt for states cannot be used directly
- Problem: incompatible state transformations
- State MaxEnt for ancilla-assisted tomography
Incomplete PT: MaxEnt

- concept of state entropy for ancilla-assisted PT
- extension to nonancilliary approach

\[ (\rho_k, A) \leftrightarrow A \otimes X_k \]

\[ \langle A \rangle_{E[\rho_k]} = \langle A \otimes X_k \rangle_{E \otimes I[\Omega]} \]

\[ \langle E^*[A] \rangle_{\rho_k} = \langle E^*[A] \otimes X_k \rangle_{\Omega} \]

\[ \sum (\rho_k)_{ab}(E^*[A])_{ba} = \sum \omega_{ab,cd}(X_k)_{dc}(E^*[A])_{ba} \]

\[ \bar{X}_k = [\Omega]^{-1} \bar{\rho}_k \]

- for max. entangled state
- problem which \( \Omega \) to use
Incomplete PT: qubit channel

MaxEnt

- 0 knowledge ... contraction to total mixture
- single measurement, i.e. \( \mathcal{O}_{\text{proc}} = \{(\rho, F)\} \)
  - data \( \rho = \frac{1}{2}I, \quad F = \vec{f} \cdot \vec{\sigma}, \quad m = \text{Tr} F \rho' \)
  - estimated channel \( \mathcal{E}_{\text{est}}[\rho] = \frac{1}{2}(I + m \vec{f} \cdot \vec{\sigma}) \)
- analytically difficult
Hypothesis testing

- problem: find **unique property** (a priori info)
- small number of measurements
- quantify validity of the hypothesis
- H1: pure state preparator $\psi$
  - test = single projective measurement
- H2: unitary transformation $U$
  - AAPT with single projective measurement
- H3: extremal channels
- H4: entanglement?
Conclusion

- complete tomography is expensive
- incomplete MaxEnt is questionable
- hypothesis testing
  - pure state verification
  - contraction to pure state
  - testing for unitaries
- “golden standards” for state and process est.
- standards for calibration
Literature

- A.Gillchrist, N.K.Langford, M.A.Nielsen, Distance measures to compare real and ideal processes, quant-ph/0408063
- and many others ...