Quantum Compilation by Optimal Control of Open Systems:
Recent Results & Perspectives for SP4

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QAP SP4 Meeting, Maria Laach, March 2007

SP4: Key Results by TUM 2006/07

- optimal control of open systems WP 4.3 & 4.5
  - Markovian example: encoded CNOT quant-ph/0609037
  - non-Markovian extension: Z-gate quant-ph/0612165
- control of coupled Josephson charge qubits WP 4.3
  - 100-fold less error for CNOT & TOFFOLI PRA 75 012302 (2007)
- local time-reversal, local optimisation WP 4.5
  - details in quant-ph/0610061, math-ph/0701035, math-ph/0702005
- parallelising GRAPE WP 4.3, 4.4, 4.5
  - 500-fold speed-up on parallel cluster LNCS 4128, 751 (2006)

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Motivation: Control in Quantum Technology

We are currently in the midst of a second quantum revolution. The first quantum revolution gave us new rules that govern physical reality. The second quantum revolution will take these rules and use them to develop new technologies. DOWLING & MILBURN, 2003

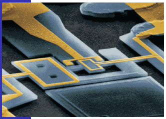
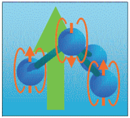
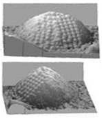
- economy

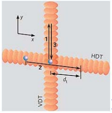
  currently some 30% of the GNP of industrial states depend on quantum effects (transistor, laser)

- technology ahead

  quantum & nano-technology rely on quantum control

  (solid-state devices, spintronics–NMR–EPR, quantum dots, ion-traps)



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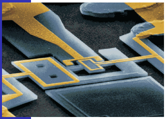
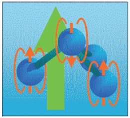
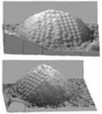
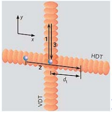
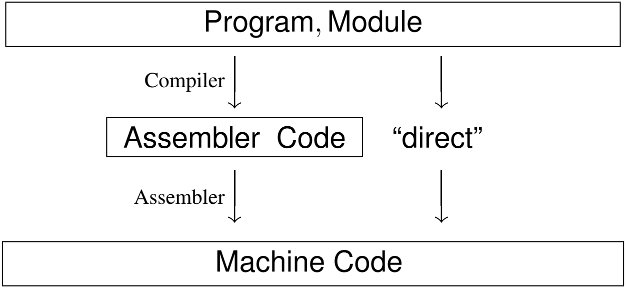






Classical Compiler
Compilation into Machine Code

- Classical Compilation



- Quantum Compilation: a Control Problem

Quantum Algorithm, Module

Precompiler ↓ ↓

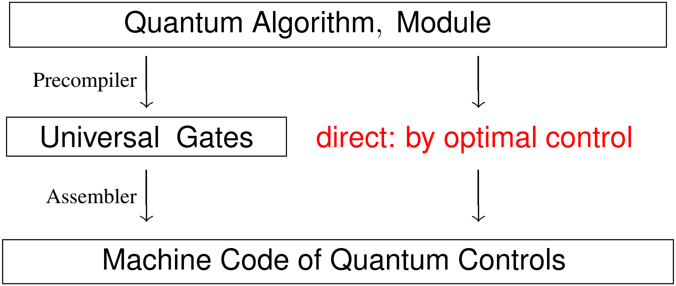
Universal Gates | direct: by optimal control

Assembler ↓ ↓

Machine Code of Quantum Controls

NB crucial for quantum machine code:
has to be timeoptimal or decoherence protected

- Quantum Compilation: a Control Problem



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Outline

- **Bilinear Control System**

$$\dot{X}(t) = \Big(A + \sum_{j=1}^{m} u_j(t) B_j\Big)\; X(t)$$

- Hamiltonian dynamics (Schrödinger equation)

$$|\dot{\psi}(t)\rangle = -i\Big(H_d + \sum_{j=1}^{m} u_j(t) H_j\Big)\; |\psi(t)\rangle$$

  - H_d: drift term
  - H_j: controls
  - $u_j(t)$: control amplitudes

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In systems of n qubits:

$$|\dot{\psi}(t)\rangle = -i\big(H_d + \Sigma_{j=1}^{m} u_j(t) H_j\big)\, |\psi(t)\rangle \in \mathbb{C}^{2^n} \qquad (1)$$

where $|\psi\rangle \in \mathbb{C}^{2^n}$ and $i\,H_\nu \in \mathfrak{su}(2^n)$.

Definition

A system is *fully controllable*, if every state on the unitary orbit can be reached (in finite time). For normal A, C this means every final state $X(t) =: C$ can be reached from any initial state $X(0) =: A$ with the same spectrum.

Corollary

The bilinear system (1) is fully controllable if drift and controls are a generating set of $\mathfrak{su}(2^n)$ by way of commutation, i.e. $\langle H_d, H_j \,|\, j = 1, 2, \dots, m\rangle_{\text{Lie}} \overset{\text{iso}}{=} \mathfrak{su}(2^n)$.

Controllability of Quantum Systems

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Controllability and Coupling Topology

- Example: n weakly coupled spins-$\frac{1}{2}$.

  Which conditions suffice for

$$\langle H_d, H_j \,|\, j = 1, 2, \ldots, m \rangle_{\mathrm{Lie}} \overset{\mathrm{iso}}{=} \mathfrak{su}(2^n) \quad ?$$

Lemma (Diss-ETH 12752)

A system of n qubits is fully controllable, if e.g. the control Hamiltonians H_j comprise $\{\sigma_{kx}, \sigma_{ky} \,|\, k = 1, 2, \ldots n\}$ on every single qubit selectively and the drift Hamiltonian H_d encompasses the Ising pair interactions $\{J_{k\ell}\,(\sigma_{kz} \otimes \sigma_{\ell z})/2 \,|\, k < \ell = 2, \ldots n\}$, where the coupling topology of $J_{k\ell} \neq 0$ may take the form of any connected graph.

Controllability and Quantum Gates

Corollary

The following are equivalent:

1. *in a quantum system drift H_d and controls H_j form a generating set of $\mathfrak{su}(2^n)$;*
2. *every unitary transformation in $SU(2^n)$ can be realised on that quantum hardware;*
3. *there is a set of universal quantum gates for the quantum system;*

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Principles: Optimal Quantum Control

Scope in Optimal Control:
maximise quality function **subject to** equation of motion

Scenarios:

- Hamiltonian dynamics

  notation: $U := e^{-itH}$; $\mathrm{Ad}_U(\cdot) := U(\cdot)U^{-1}$; $\mathrm{ad}_H(\cdot) := [H, \cdot\,]$

  1. pure state $\quad |\dot{\psi}\rangle = -iH\,|\psi\rangle \quad \in \mathcal{H}$
  2. gate $\quad \dot{U} = -iH\,U \quad \in \mathcal{U}(\mathcal{H})$
  3. non-pure state $\quad \dot{\rho} = -i\,\mathrm{ad}_H\,(\rho) \quad \in \mathcal{B}_1(\mathcal{H})$
  4. projective gate $\quad \dot{\mathrm{Ad}}_U = -i\,\mathrm{ad}_H \circ \mathrm{Ad}_U \quad \in \mathcal{U}(\mathcal{B}_1(\mathcal{H}))$

- Master equations of dissipative dynamics

  3'. non-pure state $\quad \dot{\rho} = -(i\,\mathrm{ad}_H + \Gamma)\,(\rho)$

  4'. contractive map $\quad \dot{\chi} = -(i\,\mathrm{ad}_H + \Gamma) \circ \chi \in \mathcal{GL}(\mathcal{B}_1(\mathcal{H}))$

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Typical Tasks in Optimal Quantum Control

1. **Maximising Experimental Sensitivity:**
   maximise transfer amplitude $f := |\operatorname{tr}\{C^\dagger UAU^\dagger\}|$,
   subject to equation of motion $\dot{A} = -i[H, A]$
   Glaser, T.S.H., Sieveking, Schedletzky, Nielsen, Sørensen, Griesinger, *Science* **280** (1998), 421

2. **Realise Quantum Gate U_G in Minimal Time:**
   maximise fidelity $\operatorname{Re}\operatorname{tr}\{\operatorname{Ad}^\dagger_{U_G} \operatorname{Ad}_U(T)\}$
   subject to equation of motion
   $\dot{\operatorname{Ad}}_U(t) = -i \operatorname{ad}_H \circ \operatorname{Ad}_U(t)$
   T.S.H., Spörl, Khaneja, Glaser, *Phys. Rev. A* **72** (2005), 042331

3. **Realise Module U_G with Minimal Relaxative Loss:**
   maximise fidelity $\operatorname{Re}\operatorname{tr}\{\operatorname{Ad}^\dagger_{U_G} \chi(T)\}$
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   $\dot{\chi}(t) = -(i \operatorname{ad}_H + \Gamma) \circ \chi(t)$
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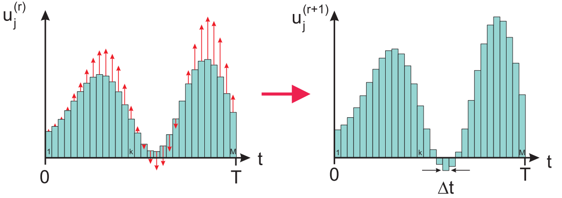
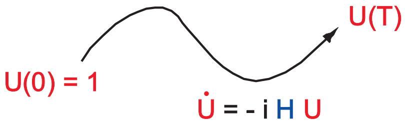
Gradient Ascent Algorithm GRAPE



J. Magn. Reson. **172** (2005), 296 and *Phys. Rev. A* **72** (2005), 042331

Generation of Unitary Operators



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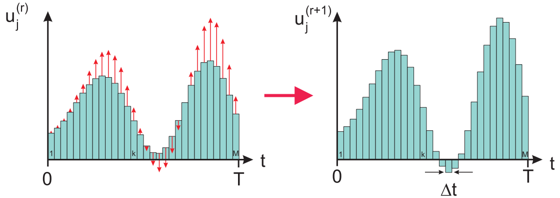

1. Define scalar-valued HAMILTON function

$$h(U) = \operatorname{Re}\operatorname{tr}\{\lambda^\dagger(-i(H_d + \textstyle\sum_j u_j H_j))U\}$$

2. with adjoint system satisfying

$$\dot{\lambda}(t) = -i(H_d + \textstyle\sum_j u_j H_j)\lambda(t) \quad .$$

3. Then PONTRYAGIN's maximum principle requires

$$\frac{\partial h}{\partial u_j} = \operatorname{Re}\operatorname{tr}\{\lambda^\dagger(-iH_j)U\} \stackrel{!}{=} 0$$

4. thus allowing for a gradient-flow of quantum controls

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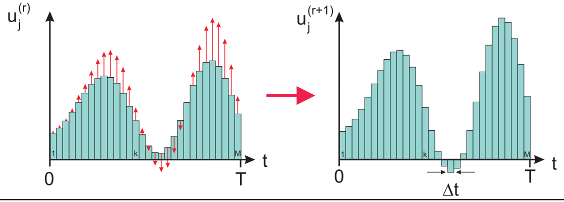

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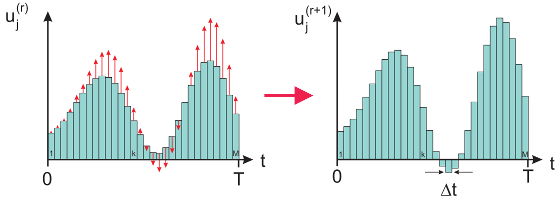
I. Quantum Compilation
II. Quantum Control
Optimising on $PSU(N)$
Decoherence Control
SP4: Dissemination
SP4: Perspectives
III. Local Control
Conclusions & Outlook

Gradient Ascent Algorithm GRAPE



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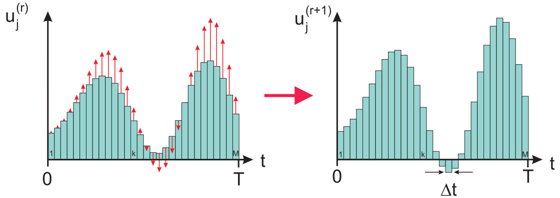

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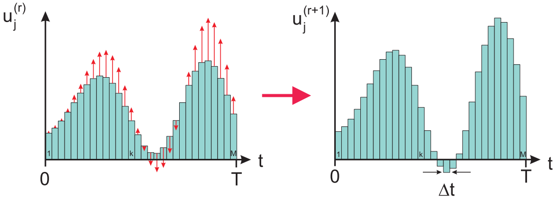

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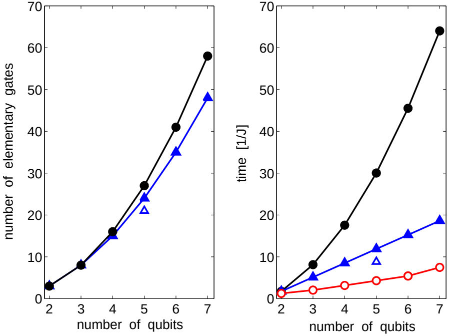
Goal: *time complexity* of QFT on linear spin chain for

- Shor's algorithm
- all algorithms of (abelian) *hidden subgroup* type

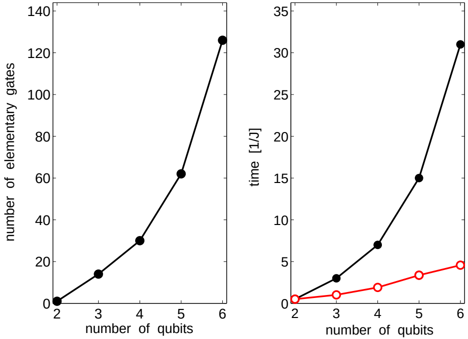


⇒ timeopt. QFT: some 3 times faster than fastest current standard-gate decompositions

Phys. Rev. A **72** (2005), 042331

Goal: *time complexity* of C^nNOT as module in

- Quantum Error Correction



⇒ timeopt. C^nNOT: quadratically faster than fastest current standard-gate decomposition

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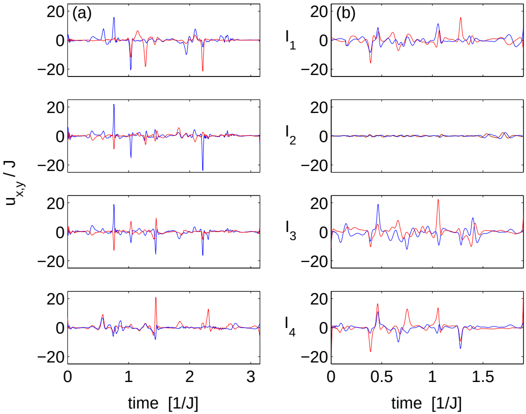
- How do 'optimal controls' look like?



$\Rightarrow$ often difficult to understand!

Goal: realise *timeoptimal* CNOT on 2 coupled charge qubits

- pseudospin Hamiltonian: $H = H_{\text{drift}}$

$$
\begin{aligned}
H_{\text{drift}} = &-\left(\frac{E_m}{4} + \frac{E_{c1}}{2}\right)(\sigma_z^{(1)} \otimes \mathbb{1}) - \frac{E_{J1}}{2}(\sigma_x^{(1)} \otimes \mathbb{1}) \\
&-\left(\frac{E_m}{4} + \frac{E_{c2}}{2}\right)(\mathbb{1} \otimes \sigma_z^{(2)}) - \frac{E_{J2}}{2}(\mathbb{1} \otimes \sigma_x^{(2)}) \\
&+ \frac{E_m}{4}(\sigma_z^{(1)} \otimes \sigma_z^{(2)})
\end{aligned}
$$

Goal: realise *timeoptimal* CNOT on 2 coupled charge qubits

- pseudospin Hamiltonian: $H = H_{\mathrm{drift}} + H_{\mathrm{control}}$

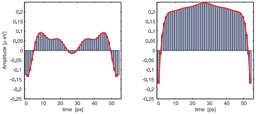
$$H_{\mathrm{drift}} = -\left(\frac{E_m}{4} + \frac{E_{c1}}{2}\right)(\sigma_z^{(1)} \otimes \mathbb{1}) - \frac{E_{J1}}{2}(\sigma_x^{(1)} \otimes \mathbb{1}) - \left(\frac{E_m}{4} + \frac{E_{c2}}{2}\right)(\mathbb{1} \otimes \sigma_z^{(2)}) - \frac{E_{J2}}{2}(\mathbb{1} \otimes \sigma_x^{(2)}) + \frac{E_m}{4}(\sigma_z^{(1)} \otimes \sigma_z^{(2)})$$

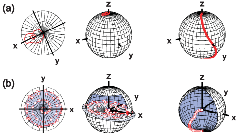
$$H_{\mathrm{control}} = \left(\frac{E_m}{2} n_{g2} + E_{c1} n_{g1}\right)(\sigma_z^{(1)} \otimes \mathbb{1}) + \left(\frac{E_m}{2} n_{g1} + E_{c2} n_{g2}\right)(\mathbb{1} \otimes \sigma_z^{(2)})$$

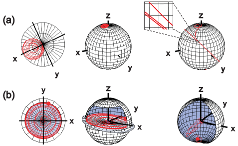
NB: components of $\{H_d, H_d, H_c\}$ form minimal generating set of $\mathfrak{su}(4)$.

Examples of Quantum Control

2. Realising Quantum Gates in Minimal Time





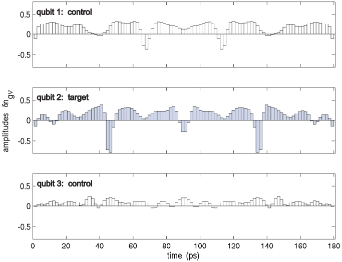


$\Rightarrow$ timeopt. CNOT: some 5 times faster than NEC group

- Quality $q := Fe^{-\tau_{\mathrm{op}}/\tau_Q}$
  so $1 - q = 1 - 0.999999999\, e^{-55\mathrm{ps}/10\mathrm{ns}} = 0.0055$
  (NEC: $1 - q = 1 - 0.4188\, e^{-250\mathrm{ps}/10\mathrm{ns}} = 0.5917$)

PRA **75**, 012302 (2007)

Goal: TOFFOLI gate on 3 linearly coupled charge qubits



13 times faster than NEC

- error rates cut by two orders of magnitude ($T_2 \simeq 10$ ns):
  1. direct gate by optimal control
     $$1 - q = 1 - 0.99999\ e^{-180\mathrm{ps}/10\mathrm{ns}} = 0.0178$$
  2. by 9 CNOT's from optimal control
     $$1 - q = 1 - (0.999999999\ e^{-55\mathrm{ps}/10\mathrm{ns}})^9 = 0.0483$$
  3. by 9 CNOT's under pioneering controls
     $$1 - q = 1 - (0.4188\ e^{-250\mathrm{ps}/10\mathrm{ns}})^9 = 0.9997$$

Examples of Quantum Control

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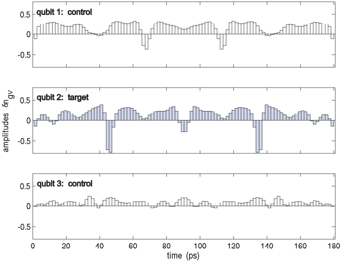

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Examples of Quantum Control

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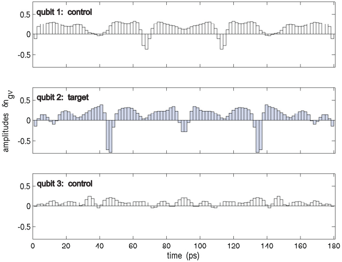

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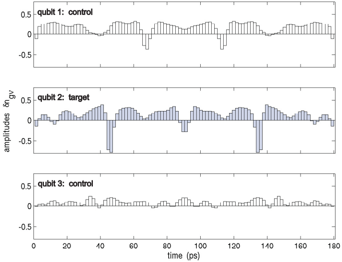

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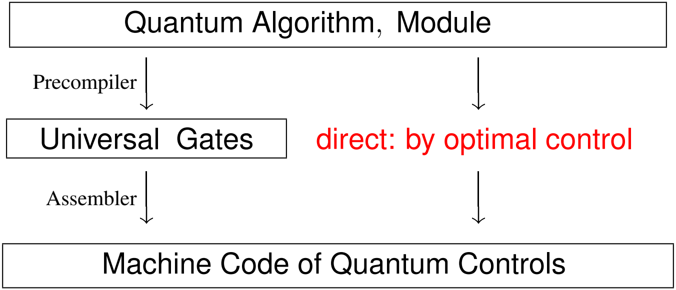
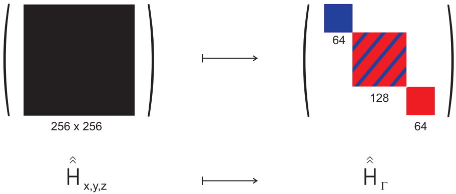
- Challenge in Quantum Compilation:
  fight decoherence by (i) timeoptimal or (ii) decoherence-protected controls!



Principle:

Code *logical* qubits in decoherence-free *physical* levels

- Master equation: $\dot{\rho} = -(i\,\mathrm{ad}_H + \Gamma)\,\rho$
- DFS: eigenspace to Γ with eigenvalue =0
- Express $\widehat{H} \equiv \mathrm{ad}_H$ in eigenbasis of Γ (here 4 qubits)



- Idea: perform calculation (e.g. CNOT) within DFS

Zanardi, Rasetti, *Phys. Rev. Lett.* **79** (1997), 3309.

Lidar, Chuang, Whaley, *Phys. Rev. Lett.* **81** (1998), 2594.

Examples of Quantum Control

3. Idea of Decoherence-Free Subspaces (DFS)

Principle:

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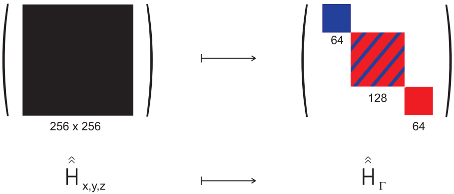

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- 1 logical qubit coded by 2 physical qubits in Bell states

$$|0\rangle_L := |\psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\ ,\ |1\rangle_L := |\psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\mathcal{B} := \text{span}\,\{|\psi^\pm\rangle\langle\psi^\pm|, |\psi^\mp\rangle\langle\psi^\pm|\}$$

Examples of Quantum Control

3. Model System: 2 Qubits by 4 Spins

- 1 logical qubit coded by 2 physical qubits in Bell states

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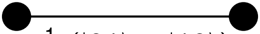
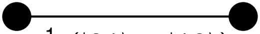
- 2 logical qubits coded by 4 physical qubits

$$\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$



$$\tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$



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- 2 logical qubits coded by 4 physical qubits

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$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- protection against T_2 relaxation (Redfield: $\Gamma \sim [ZZ, [ZZ, \rho]]$)

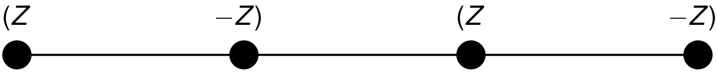
because $[\rho, ZZ] = 0 \quad \forall \quad \rho \in \mathcal{B} \otimes \mathcal{B}$

- controls

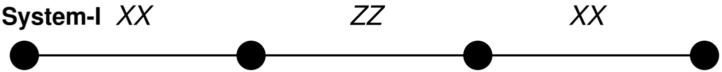
Examples of Quantum Control

3. Model Systems: 4 Linearly Coupled Spins

- controls



- drift: Ising (ZZ) and Heisenberg (XX) interactions



Examples of Quantum Control

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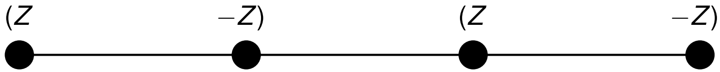

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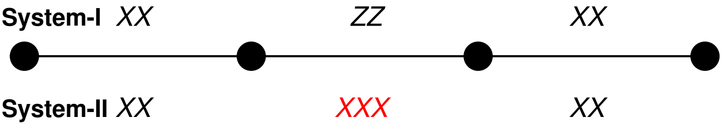
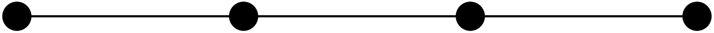


Examples of Quantum Control

3. Model Systems: 4 Linearly Coupled Spins

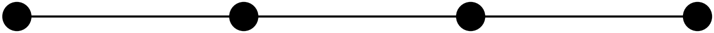
- controls

$(Z \quad -Z) \quad (Z \quad -Z)$



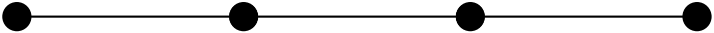
- drift: Ising (ZZ) and Heisenberg (XX,XXX) interactions

System-I $XX \quad ZZ \quad XX$



System-II (2 Hz) XX (1 Hz) XXX (2 Hz) XX

- relaxation ($T_2^{-1} : T_1^{-1}$ = $4.0\ \mathrm{s}^{-1} : 0.024\ \mathrm{s}^{-1}$ $\simeq 170 : 1$)



$T_2 : \; [ZZ, [ZZ, (\cdot)]] \qquad [ZZ, [ZZ, (\cdot)]]$

$T_1 : \; [A_{2,\bullet}, [A^{\dagger}_{2,\bullet}, (\cdot)]] \qquad [A_{2,\bullet}, [A^{\dagger}_{2,\bullet}, (\cdot)]]$

System-I: staying within slowly-relaxing subspace

- drift Hamiltonian D_1 with Ising-ZZ
- controls $C_{1,2}$

$$\begin{aligned} D_1 &:= J_{xx}\,(xx\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}xx + yy\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}yy) + J_{zz}\,\mathbb{1}zz\mathbb{1} \\ C_1 &:= z\mathbb{1}\mathbb{1}\mathbb{1} - \mathbb{1}z\mathbb{1}\mathbb{1} \\ C_2 &:= \mathbb{1}\mathbb{1}z\mathbb{1} - \mathbb{1}\mathbb{1}\mathbb{1}z\,. \\ &\Rightarrow \langle D_1, C_1, C_2\rangle_{\text{Lie}}\big|_{\mathcal{B}\otimes\mathcal{B}} \stackrel{\text{iso}}{=} \mathfrak{su}(4) \end{aligned}$$

- Liouville subspace $\mathcal{B}\otimes\mathcal{B}$
  spans states protected against T_2-relaxation

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$$

  - Liouville subspace $\mathcal{B}\otimes\mathcal{B}$ spans states protected against T_2-relaxation

- **System-II**: driving outside slowly-relaxing subspace
  - drift: extended to isotropic Heisenberg-XXX

$$D_1 + D_2 \;:=\; J_{xx}\left(xx\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}xx + yy\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}yy\right) + J_{xyz}\left(\mathbb{1}xx\mathbb{1} + \mathbb{1}yy\mathbb{1} + \mathbb{1}zz\mathbb{1}\right)$$

  - Lie-algebraic closure: in 66-dim. Lie algebra

$$\dim\langle (D_1 + D_2), C_1, C_2\rangle_{\mathrm{Lie}} = 66\,,$$

  - $\mathfrak{su}(4)$ merely subalgebra

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System-II:

- full controllability within slowly-relaxing subspace
  - observation

$$e^{-i\pi C_1}(D_1 + D_2)e^{i\pi C_1} = D_1 - D_2$$

  - Trotter limit

$$\lim_{n\to\infty}\left(e^{-i(D_1+D_2)/(2n)}e^{-i(D_1-D_2)/(2n)}\right)^n = e^{-iD_1}$$

  - reduction of dynamics

$$\textbf{System-II} \xrightarrow{\text{infinite \# switchings}} \textbf{System-I}$$

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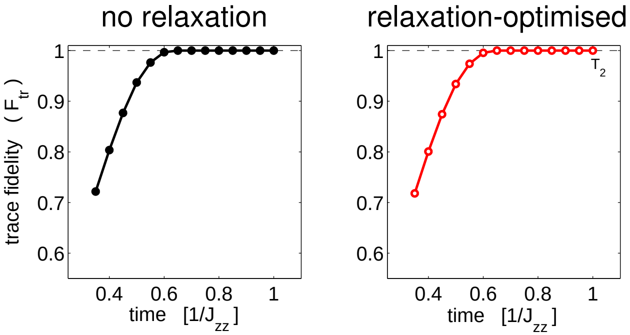

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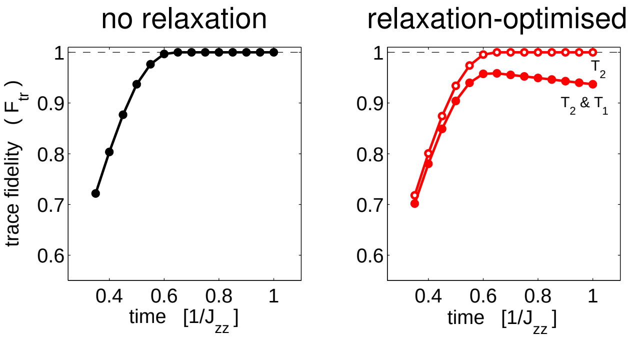
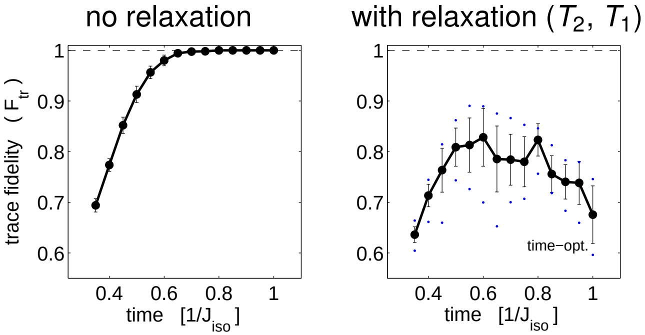


- T_2-relaxation has no effect on quality
- additional T_1-relaxation drops quality

- **System-II**: driving outside slowly-relaxing subspace



- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently

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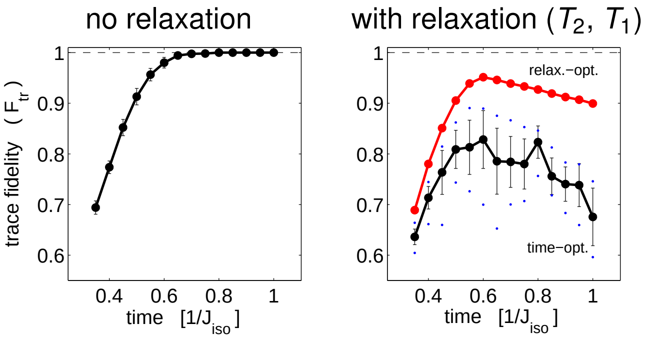
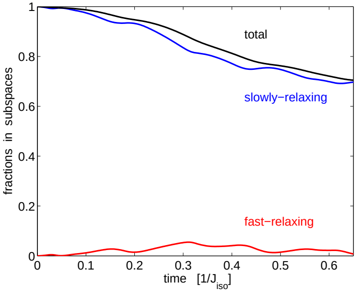


- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently
- relaxation-optimised: systematic substantial gain

quant-ph/0609037

CNOT under **System-II**: Projection into Subspaces

■ time-optimal



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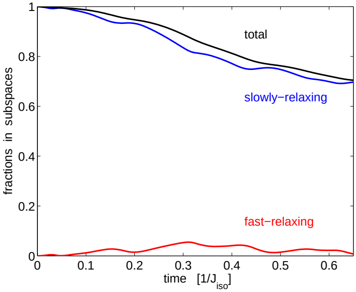

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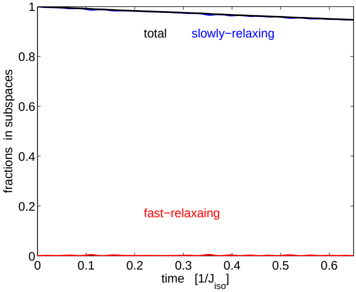

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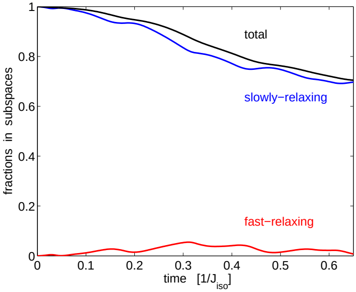

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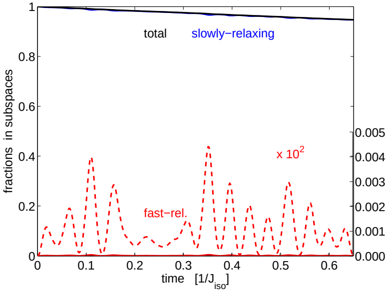
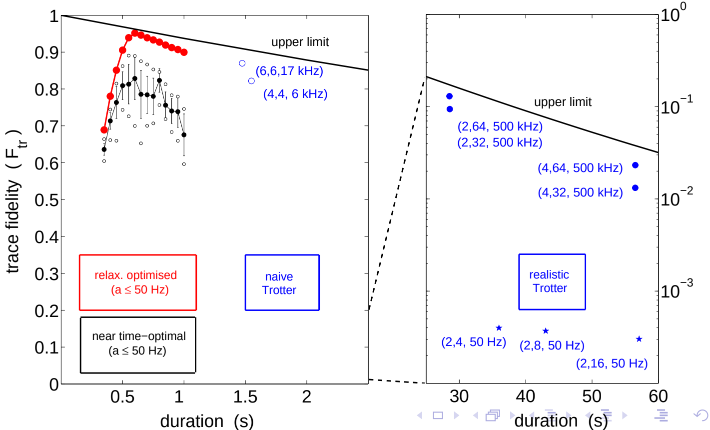


quant-ph/0609037

CNOT under **System-II**: comparison of methods

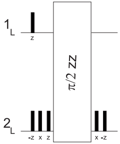
by decoherence control: $>$ 95% fideltity

conventional: $<$ 15% fidelity



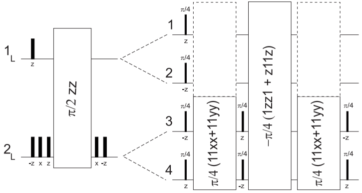
Decoherence-Protected CNOT-Gate via

- logical qubits



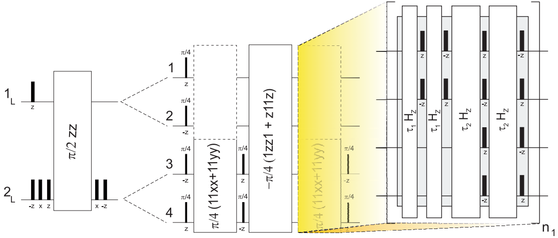
Decoherence-Protected CNOT-Gate via

- physical qubits



Decoherence-Protected CNOT-Gate via

- realisation by **System-I**



Decoherence-Protected CNOT-Gate via

- realisation by **System-II**

Which Tool in Which Setting?

1. *"anything goes"*: Paul FEYERABEND
   only in ideal case: decoherence-free, fully controllable and closed under drift

2. *Timeoptimal Control*:
   whenever slowly-relaxing subsystem controllable and closed under drift

3. *Relaxation-Optimised Control*:
   whenever slowly-relaxing subsystem open, where subsystem
   (i) controllable or
   (ii) to be extended for controllability

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3. *Relaxation-Optimised Control*:
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Which Tool in Which Setting?

1. *"anything goes"*: Paul FEYERABEND
   only in ideal case: decoherence-free, fully controllable and closed under drift

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How to run the package in MATLAB.

1. Setup up system:
   - choose example
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     `setup_your_system`
   - specify target unitary gate
2. Run `octane.m` for fixed final time T:
   - use output of the setup as input
     type `help octane` for instructions
3. Find minimum time by tracking:
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Checks for `MATLAB` (run in faster non-JAVA version).

- worked example A: CNOT in 2 qubits
  - run `setup_NMR_2spins`
  - run `octane(32,7e3,1,ones(1,4),.5,'myPulse.mat')`
  - test fidelity 0.99963 in 1489s (Pentium III 933.377MHz)
- worked example B: TOFFOLI in 3 qubits
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  - run `octane(64,1e4,1,ones(1,6),2,'myPulse.mat')`
  - test fideltity 0.99988
  - in 5419s on an Intel(R) Pentium(R) 4 CPU 2.66GHz
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SP4: Joint Perspective for Open Systems

- **Starting Point:** quant-ph/0609037
  - optimal control of Markovian dissipative systems
    $\dot{\rho} = -i[H,\rho] + \Gamma\rho = -i[H,\rho] + \frac{1}{2}\sum_j [L_j, \rho L_j^\dagger] + [L_j\rho, L_j^\dagger]$
- **Experimental Systems:** Quantum Process Tomography
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New Journal of Physics
The open-access journal for physics

Quantum process tomography and Linblad estimation of a solid-state qubit

M Howard[1], J Twamley[2,4], C Wittmann[3], T Gaebel[3], F Jelezko[3] and J Wrachtrup[3]

Definition (math-ph/0701037, math-ph/0702005)

The *local C-numerical range* is the set

$$W_{\mathrm{loc}}(C, A) := \{\mathrm{tr}\,(C^\dagger UAU^\dagger) \mid U \in SU(2)^{\otimes n}\} \subseteq W_C(A),$$

where the unitary orbit is restricted to *local operations* $U =: K \in SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$

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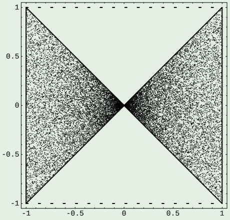
Example (non convex)

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

$$C := \text{diag}\,(1, 0, 0, 0)$$



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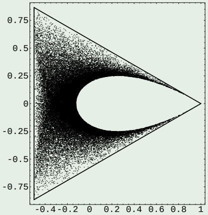
Example (neither star-shaped nor simply connected)

$$A := \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi/3} \end{pmatrix}^{\otimes 3}$$

$$C := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes 3}$$



- Gradient flow on *local* unitaries

$$\dot{K} = \operatorname{grad} f(K) = P_{\mathfrak{k}}([(KHK^{-1}), H])\, K$$
$$= -P_{\mathfrak{k}}\big(\mathrm{ad}_{\mathrm{H}} \circ \mathrm{Ad}_{\mathrm{K}}(H)\big)\, K\,,$$

$P_{\mathfrak{k}}$: projection onto subalgebra $\mathfrak{k}$ of generators of local unitaries $\mathbf{K} = \mathbf{SU}(2)^{\otimes n}$.

Applications

Pure-State Entanglement

math-ph/0701037

- Maximising real part in $W_{\rm loc}(C,A)$ minimises distance from C to *local unitary orbit* of A

$$\max_{K\in SU(2)^{\otimes n}} \operatorname{Re}\operatorname{tr}\{C^\dagger KAK^{-1}\} \Leftrightarrow \min_{K\in SU(2)^{\otimes n}} \|KAK^{-1}-C\|_2$$

- Application to Quantum Information Theory: let A be a given rank-1 state of the form $A=|\psi\rangle\langle\psi|$ and $C=\operatorname{diag}(1,0)^{\otimes n}$ [thus $W_{\rm loc}(C,A)\to W_{\rm loc}(A)$]

Corollary (Interpretation)

The minimial Euclidean distance is a measure of (pure-state) entanglement; i.e. it quantifies how far A is from the local equivalence class of the tensor-product state C.

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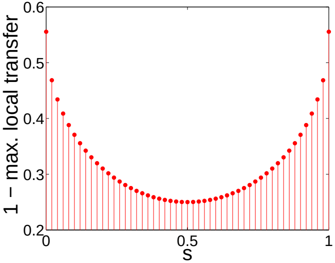
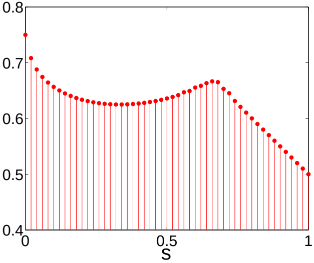
The minimial Euclidean distance is a measure of (pure-state) entanglement; i.e. it quantifies how far A is from the local equivalence class of the tensor-product state C.

- Examples:
  pure-state entanglement parameterised by s

$|\psi_3(s)\rangle = \sqrt{s}|W\rangle + \sqrt{1-s}|\tilde{W}\rangle$



$|\psi_4(s)\rangle = \sqrt{s}|GHZ'\rangle - \sqrt{1-s}|\psi_+\rangle \otimes |\psi_+\rangle$



- CPU times: local gradient flows
  very fast as compared to global techniques

- Example 1: distance to 3-qubit *W* state
- Example 2: distance to 4-qubit GHZ-type state

qubits	polynomial optim. cpu-time [sec][1]	by gradient flow cpu-time [sec][2]	speed-up
3	10.92	0.30	36.4
4	103.97	0.71	147.0

[1] Eisert *et al.* (processor with 2.2 GHz, 1 GB RAM)
[2] average of 50 runs, Athlon XP1800+ (1.1 GHz, 512 MB RAM)

Applications

Local Time Reversal

quant-ph/0610061

- Task: time reversal by *local* operations

  Decide whether sign-reversed Hamiltonian $-H$ is on the *local unitary orbit* of the original Hamiltonian H!
  $? \, \exists \, K \in SU(2)^{\otimes n} \; : \; K \, e^{-itH} \, K = e^{+itH}$

Corollary (Local time reversal)

For $H = H^\dagger$ with $\|H\|_2 = 1$ the following are equivalent:

1. *the Hamiltonian H is locally sign-reversible;*
2. *its local C-numerical range comprises -1;*
   $-1 \in W_{\mathrm{loc}}(H, H)$
3. $\exists K \in SU(2)^{\otimes n}$:
   $\|KHK^{-1} + H\|_2^2 = 0 \;\Leftrightarrow\; \mathrm{Ad}_K(H) = -H$

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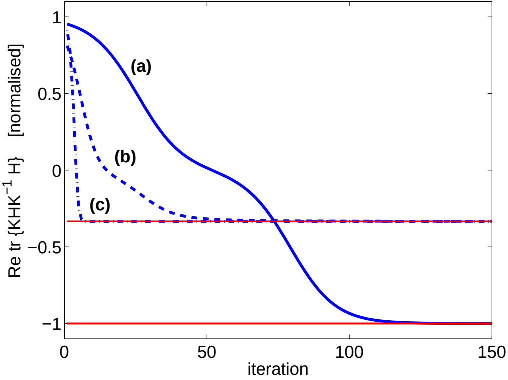
- Examples
  (a) ISING ZZ-interaction on cyclic graph C_4 (bipartite)
  (b) ISING ZZ-interaction on cyclic graph C_3 (not bipartite)
  (c) HEISENBERG XXX interaction (isotropic coupling)



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Corollary (Extension II: Lie algebras)

For $H = H^\dagger$ with $\|H\|_2 = 1$ the following are equivalent:

1. *the Hamiltonian H is locally sign-reversible;*
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3. *H is locally unitarily similar to a $\bar{H}$ with $\mathrm{Ad}_{K_z}(\bar{H}) = -\bar{H}$;*
4. *let $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} \mathbb{C} E_{ij}$ be the root-space decomposition of $\mathfrak{sl}(N, \mathbb{C})$; H is locally unitarily similar to a linear combination of root-space elements to non-zero roots $\bar{H} := \sum_{\lambda=1}^{m} c_\lambda E_{ij}^{(\lambda)}$ satisfying a system of linear equations $\sum_\ell p_{\lambda,\ell} \cdot \phi_\ell = \pi (\mathrm{mod}\, 2\pi)$*

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   - grossly fighting decoherence
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   - new: local C-numerical range

Conclusions

"Show me the big effects!"

1. Quantum Control: key in future technology
2. Timeoptimised Quantum Compilation
   - dressed to physical hardware
   - access to time complexity
   - grossly fighting decoherence
3. Decoherence-Minimising Quantum Control
   - progress towards controllability under dissipation
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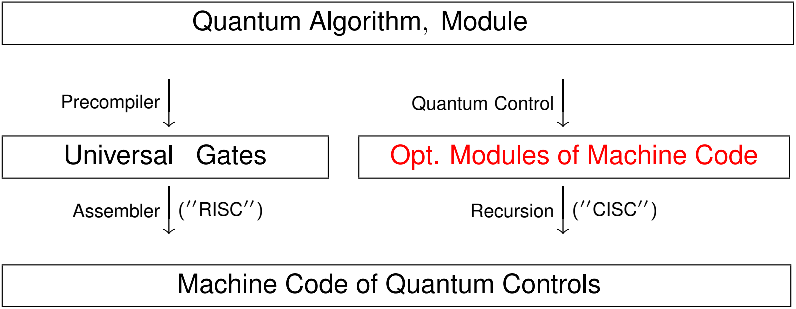
Outlook

Compilation by Recursion for Large Systems

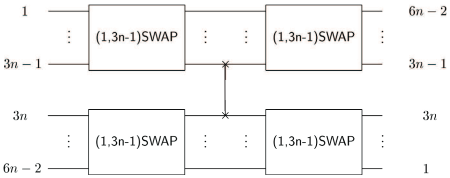
- recursive use of optimised medium-sized building blocks



Goal: $(1, 6n)$ SWAP

Principal Ways:

1. via $(1, 3n)$



2. via $(1, 2n)$

Goal: $(1, 6n)$ SWAP

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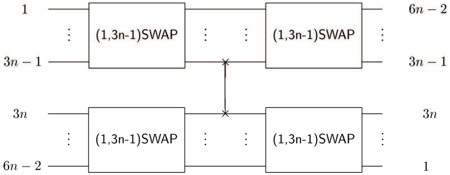
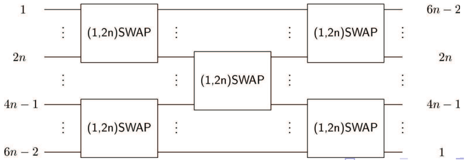
1. via $(1, 3n)$

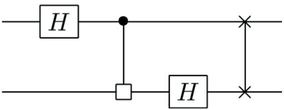


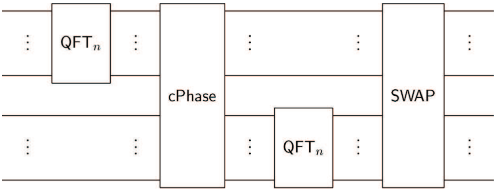
2. via $(1, 2n)$

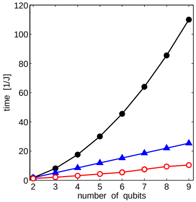


I. Quantum Compilation
II. Quantum Control
SP4: Dissemination
SP4: Perspectives
III. Local Control
Conclusions & Outlook

- **Principle:** from 2-qubit QFT to 2n-qubit QFT





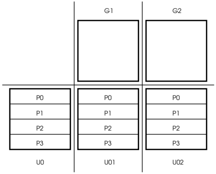


Parallelisation

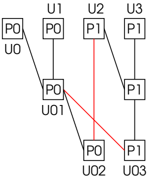
Speed-Up on High-Performance Parallel Cluster with T. Gradl, T. Huckle

- Parallelising Matrix Operations

  1. slice-wise:



  2. tree-like:



- Resulting Speed-Ups: 10 spins 128 time slices

128 AMD *Opteron 850* CPU (2.4 GHz)

Subroutine	% of time	Speedup
optimizeCG	100	578
maxStepSize	90	709
getGradient	9.1	187
expm	7.5	879
propagation	1	31
gradient	0.6	81

Gradl, Spörl, Huckle, Glaser, T.S.H., *Proceedings EUROPAR LNCS* **4128** (2006), 751

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Steffen J. Glaser

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References:

J. Magn. Reson. **172**, 296 (2005), *PRA* **72**, 043221 (2005),
EUROPAR Lect. Notes Comput. Sci. **4128**, 751 (2006), *PRA* **75**, 012302 (2007),
quant-ph/0609037, quant-ph/0610061, quant-ph/0612165,
math-ph/0701035, math-ph/0702005